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A COMPARISON BETWEEN FREQUENCY
MODULATION AND AMPLITUDE MODULATION
FOR A DISTANCE MEASURING EQUIPMENT

JOHN N. MORRISSEY

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A COMPARISON BETWEEN FREQUENCY MODULATION
AND AMPLITUDE MODULATION FOR A
DISTANCE MEASURING EQUIPMENT

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John N. Morrissey

A COMPARISON BETWEEN FREQUENCY MODULATION
AND AMPLITUDE MODULATION FOR A
DISTANCE MEASURING EQUIPMENT

by

John N. Morrissey

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ENGINEERING ELECTRONICS

United States Naval Postgraduate School
Monterey, California

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This work is accepted as fulfilling
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MASTER OF SCIENCE

IN

ENGINEERING ELECTRONICS

from the

United States Naval Postgraduate School

ABSTRACT

Missiles of various types have become operational in the Armed Forces with the current emphasis shifting to the long-range ballistic missile. To study the characteristics of these missiles in flight, a method of accurately positioning the missile is necessary to obtain the desired scientific data. The Cubic Corporation of San Diego, California has developed an accurate system for determining missile position which is presently operational for short-range missiles but the need exists to extend the range to the 2000 mile magnitude or greater to handle the ballistic missile. The prime purpose of obtaining scientific data dictates a high degree of accuracy for this system. A theoretical comparison in performance between amplitude modulation and frequency modulation is made for the distance-measuring equipment of the Cubic ranging system. This comparison includes not only the present operational system, but also the proposed methods of extending system range by removing the present threshold level through correlation techniques.

The writer wishes to express his appreciation for the assistance and encouragement given him by Professors C. F. Klammer, Jr. and P. E. Cooper of the U. S. Naval Postgraduate School in this investigation.

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1. Introduction

The Cubic Corporation of San Diego, California has produced an operational electronic distance-measuring equipment (DME) which is used as ranging equipment at missile test sites. It is normally incorporated with two angle-measuring equipments (AME) into systems known as COTAR and MOPTAR which by the simple geometry of measuring two angles and a range accurately from fixed positions on the ground determine missile position. The continuous measurement of missile position provides a missile tracking capability. The MOPTAR system, in addition, uses time multiplex principles to obtain a multiple target tracking capability of three missiles. Although only the DME section of the above systems will be discussed, it is pertinent to note that the AME section operates on similar basic principles and many of the results obtained here apply equally as well to the AME.

The basic DME principle is based on the fact that the modulation of a modulated wave propagated through space experiences a phase shift proportional to the distance traveled in modulation frequency wavelengths. It is independent of the carrier frequency. The phase shift can be expressed as

$$\beta = \frac{2\pi D}{\lambda_m} \quad (1-1)$$

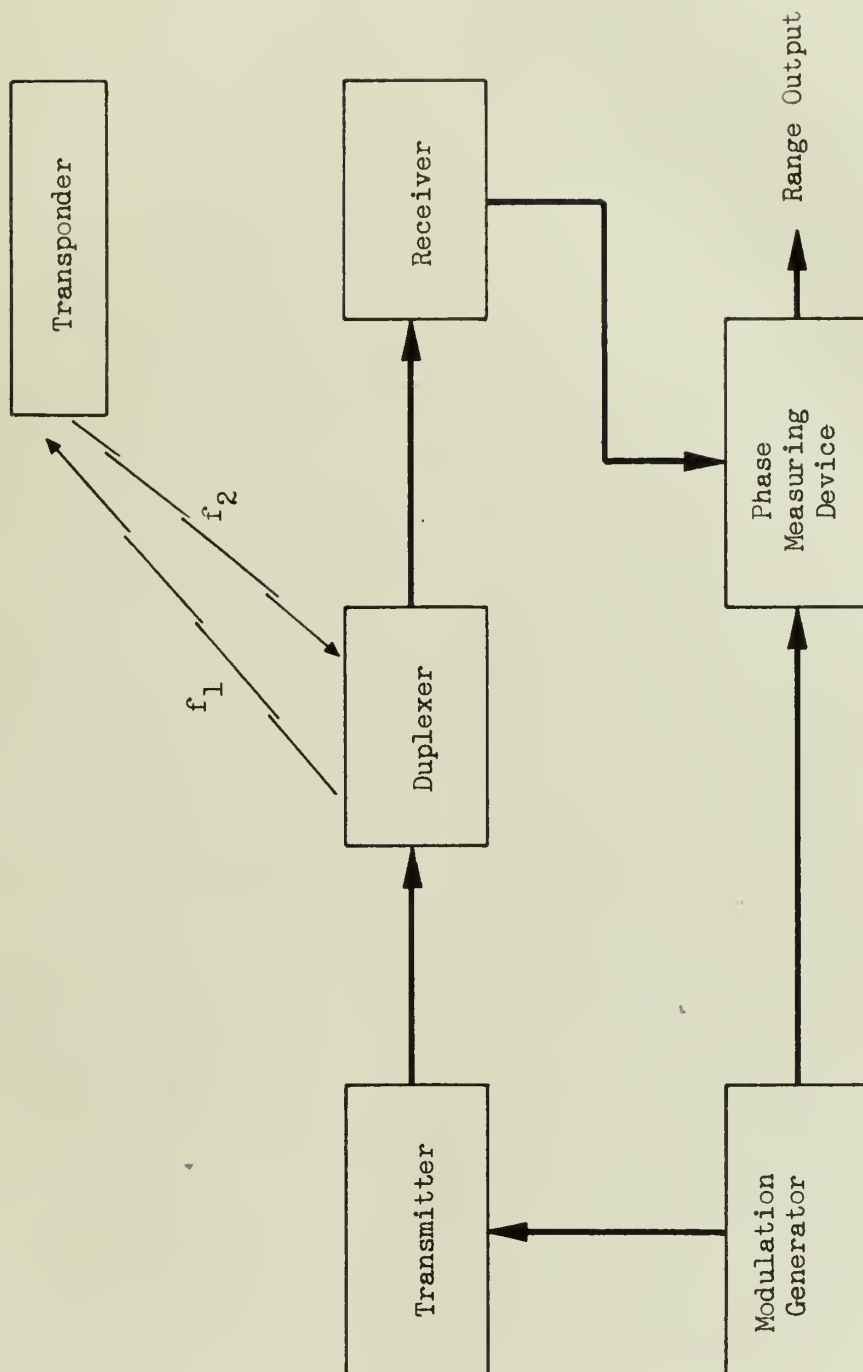
where β = phase shift in radians

D = the distance traveled

λ_m = modulation wavelength

D and λ_m in the same units

The basic DME system shown in Figure 1 consists of a ground station and a transponder located in the missile. The ground station consists of an f-m transmitter operating at carrier frequency f_1 , an f-m receiver operating at carrier frequency f_2 and a servo mechanism type of phase-



SIMPLE BLOCK DIAGRAM OF DME

Figure 1

measuring device. The transponder is an f-m receiver-transmitter which receives the f-m signal from the ground station at carrier f_1 and re-transmits the modulation at carrier f_2 . It is a constant offset type with $f_2 = f_1 \pm f_{IF}$. The transponder signal is, in turn, received by the ground station and the phase difference between the reference outgoing modulation signal and the demodulated received signal is measured to provide a data readout proportional to the distance between the measuring station and the transponder. Since the two-way range is actually measured by the system, the factor of one-half and any constant phase shift through the system are calibrated out by adjusting the servo readout system to read the true range.

The design specifications of the DME require slant range measurements over an interval from one-half to 200 miles with a maximum rms error of ± 2.5 feet and range rates up to 25,000 feet per second in this interval. An examination of the basic DME equation shows that accuracy requires a short modulation wavelength but to resolve ambiguity in the system a long modulation wavelength is required. Cubic resolves these incompatible objectives by using a total of five modulation frequencies as indicated in the table below.

CHANNEL	FREQUENCY(KC)	WAVELENGTH(ft)	SLANT RANGE(ft/cycle)
Very fine	491.76	2,000	1,000
Fine	61.47	16,000	8,000
Intermediate	7.684	128,000	64,000
Coarse	1.921	512,000	256,000
Very Coarse	.192	5,120,000	2,560,000

Data for each channel is obtained from the receiver output by filtering. Each channel is provided with a separate phase-measuring device which operates continuously so that a direct reading of slant range is always available. Accuracy is thus obtained from the very fine channel while ambiguities are resolved in the remaining channels.

The 491.76 kc modulation signal for the very fine channel is obtained from an oven controlled crystal with an accuracy of 0.0001% and the remaining modulation signals are derived from this crystal using divider circuits. It should be noted that the modulation frequencies are harmonically related for greater accuracy and that the accuracy requirements are less for each succeeding channel in the order listed in the above table.

The objective of accuracy dictates a high order of system circuit stability. Cubic has solved this problem by (1) the selection of components and circuits with the best stable phase characteristics possible, e.g., use of broad band low Q circuits to minimize the phase shift due to slight circuit detuning, (2) the use of common components where feasible and (3) the application of negative feedback techniques.

The carrier frequencies used are in the 200-500 mc range. Cubic's experience has shown that the systems built with carrier frequencies throughout this range have proven to possess the desired characteristics with respect to ground plane, multipath problems and antenna size and array features that are required for the combined AME-DME systems.

The selection of the modulation index for each modulation frequency was based on the following: (1) utilizing as narrow an energy band as possible and (2) maintaining a stable linear phase characteristic through

the IF strip over a wide range of input signal levels. A modulation index of one was chosen for the very fine channel with succeeding channels having indices of 4, 16, 128, and 256, respectively. By actually modulating the carrier with a linear sum of the five modulation frequencies, the energy band becomes approximately equivalent to a single modulation of index 2 thus meeting criteria (1) above. The second criteria is met by utilizing the negative feedback principles developed by Chaffee [6] which reduces the modulation index to an effective value of .02 and using synchronously tuned cascaded stages for the IF strip. The DME has 6 stages in IF strip with an effective bandwidth of 2 mc.

The IF strip is followed by a limiter and conventional Foster-Seely discriminator. The limiter enhances operation over a wide range of input signal levels. The discriminator detection scheme has an inherent threshold effect determined by the predetection bandwidth and for the DME occurs at a signal-to-noise ratio of approximately 10 db. This threshold represents a definite limitation to the present system.

Cubic Corporation has been building the DME for approximately 7 years. It is relatively safe to assume that present operational techniques represent the optimum in design for the circuitry employed and that improvements must come from new concepts. The MOPTAR system represents the latest improved operational system and employs two new principles. The first is phase modulation instead of frequency modulation. The second is a new modulation scheme in which the lower modulation frequencies modulate the upper two modulation frequencies and the sum of these complex waves modulates the carrier. The advantage of the latter is the elimination of the modulation difficulties inherent in modulating a high carrier frequency by a small modulation frequency plus an improved operational performance. The former gives a general improved

performance but is additionally pointed to the incorporation of the phase-lock correlation system to eliminate the present detection threshold. The correlation phase lock system has been under study by Cubic and a 20 db improvement in signal-to-noise threshold level has been attained in the breadboard stage. Cubic has also proposed an all-electronic phase meter to replace the servo unit and this meter has been tested in the breadboard stage. Both of the above two components are ready for incorporation in future systems. Proposals by C. A. Skinner [23] and W. R. Sheridan [22] in Masters theses on this system are also worthy of note. Skinner investigated the phase detector as a correlation device and actual hardware has been advanced for the phase lock correlation system as mentioned above. The synchronous detector and the concept that AM would provide considerable improvement over FM proposed by Sheridan have not been investigated. The above proposals represent the present status of the DME.

The purpose of this thesis will be to investigate fully the proposal that AM provides considerable improvement over FM and will include techniques for removing the present system threshold for future systems to obtain greater range.

2. Concept of Using AM.

The original concept of using AM instead of FM arose from the consideration of the effective modulation index of .02 in the IF strip. Considering only one modulation frequency for ease of mathematics and by the principle of superposition, it may be shown that for values of modulation index $\leq .2$ that an AM and an FM wave are essentially the same. In general an FM wave may be expressed

$$e_{fm} = E \sin (\omega_c t + m_f \sin \omega_m t) \quad (2-1)$$

By the Bessel function expansion this may be written

$$\begin{aligned} e_{fm} = & J_0(m_f) E \sin \omega_c t \\ & + J_1(m_f) E [\sin (\omega_c + \omega_m)t - \sin (\omega_c - \omega_m)t] \\ & + J_2(m_f) E [\sin (\omega_c + 2\omega_m)t + \sin (\omega_c - 2\omega_m)t] \\ & + \dots \\ & + J_n(m_f) E [\sin (\omega_c + n\omega_m)t + (-1)^n \sin (\omega_c - n\omega_m)t] \end{aligned} \quad (2-2)$$

where $m_f = \frac{\Delta\omega}{\omega_m} = \frac{\text{maximum deviation of carrier frequency}}{\text{Modulation frequency}}$

and is defined as the index of frequency modulation.

Expanding (2-1) by $\sin (x + y) = \sin x \cos y + \cos x \sin y$ gives

$$e_{fm} = E [\sin \omega_c t \cos (m_f \sin \omega_m t) + \cos \omega_c t \sin (m_f \sin \omega_m t)] \quad (2-3)$$

but $\cos (m_f \sin \omega_m t) \approx 1$ for $m_f \leq .2$

$\sin (m_f \sin \omega_m t) \approx m_f \sin \omega_m t$ for $m_f \leq .2$

since for the maximum value of $(m_f \sin \omega_m t)$ of .2

$$\cos (.2) = \cos 11^\circ 46' = .98 \approx 1$$

$$\sin (.2) = \sin 11^\circ 46' = .198 \approx \text{the argument } .2$$

Thus, for the effective value of m_f of .02 the above approximations are even more accurate and therefore equation (2-3) may be written

$$e_{fm} = E (\sin \omega_c t + m_f \sin \omega_m t \cos \omega_c t) \quad (2-4)$$

The well known expression for AM is

$$e_{am} = E (\sin \omega_c t + m_a \sin \omega_m t \sin \omega_c t) \quad (2-5)$$

It is obvious that equations (2-4) and (2-5) differ only in the second term in which the FM wave has a $\cos \omega_c t$ factor while the AM wave has a $\sin \omega_c t$ factor. Essentially this means that the AM and FM modulations are in phase quadrature with respect to the carrier. Therefore, it may be concluded that AM and FM are essentially the same for an index of modulation $\leq .2$. It should also be noted that the above conclusion is equally applicable to a PM wave in which the only change required is to replace m_f by m_p , the index of phase modulation which is defined as the maximum phase shift from the mean.

3. Basis of Comparison.

Any comparison made between two physical systems must have a fundamental standard upon which the relative merits of the two systems are based. Standards are sometimes arbitrarily selected and are often biased and misleading. The DME system, however, possesses a unique property upon which to base the standard of comparison and that is accuracy. In this system accuracy infers phase stability since phase difference is the measured quantity. To achieve the necessary phase stability, modulation feedback is considered essential to stabilize the total modulation phase delay through the receiver as a function of received signal level and carrier frequency. The source of modulation phase shift through the receiver which must be stabilized include the following:

- (1) change in the effective Q due to AGC action
- (2) change in the IF frequency due to oscillator drift
- (3) change in the tuning of the IF stages due to tube aging
- (4) change in the detector phase transfer characteristics due to driving level variations
- (5) phase shifts caused by intermodulation distortion of strong signals applied to the mixer
- (6) changes in the tuning of the low pass and bandpass networks due to temperature drift
- (7) change in the supply voltage variations causing a change in g_m .

Experimental results obtained by Cubic and contained in the Final Engineering Report DME [7] show that modulation feedback will provide the necessary phase stability when an effective feedback loop gain of 25 to 30 db is provided at each modulation frequency. Conventional negative feedback principles will not provide this stability since the amount

of feedback is proportional to the input signal level and the dynamic input range required is too large. In addition the desired IF gain of 90 db complicates this problem because with conventional negative feedback, the gain per stage is effectively reduced by the amount of feedback employed and there is a limit to the number of stages that may be utilized in cascade. More basically conventional negative feedback stabilizes the amplifier and not the modulation. To compare the AM and FM systems it is necessary to first analyze the modulation feedback loop for both systems in detail. The results of these analyses then form the foundation for comparison of the second order modulation effects, AGC, power, range and detection schemes. It will also become evident that the principles of modulation feedback are detrimental to the AM technique since it restricts the effective AM modulation index to a very small value and generates a large second harmonic distortion term. However, it has been shown above that modulation feedback is required to obtain the necessary phase stability. Conversely for FM the modulation feedback technique enhances this type modulation since it not only provides the necessary phase stability but does this without distortion.



4. Modulation Feedback Loop.

The basic modulation feedback loops for FM and AM are shown in Figures 2 and 3, respectively. There are two important considerations with regard to the action of the loops: (1) to prove that the input to the IF strip is actually an FM or AM wave and (2) to show that the phase stability afforded by the loop is the same for both AM and FM. Since the results of the loop action must be understood to make the comparisons between the two systems, it is considered necessary to completely derive the above considerations in detail. By the principle of superposition and for ease in handling the mathematics, only one modulating frequency will be considered throughout this paper.

A. FM Modulation Feedback Loop Analysis (Refer to Figure 2).

$$e_{fm_1} = E_1 \sin (\omega_c t + m_{f_1} \sin \omega_m t) \quad (4-1)$$

$$e_{fm_2} = E_2 \sin (\omega_o t + m_{f_2} \sin [\omega_m t + \theta]) \quad (4-2)$$

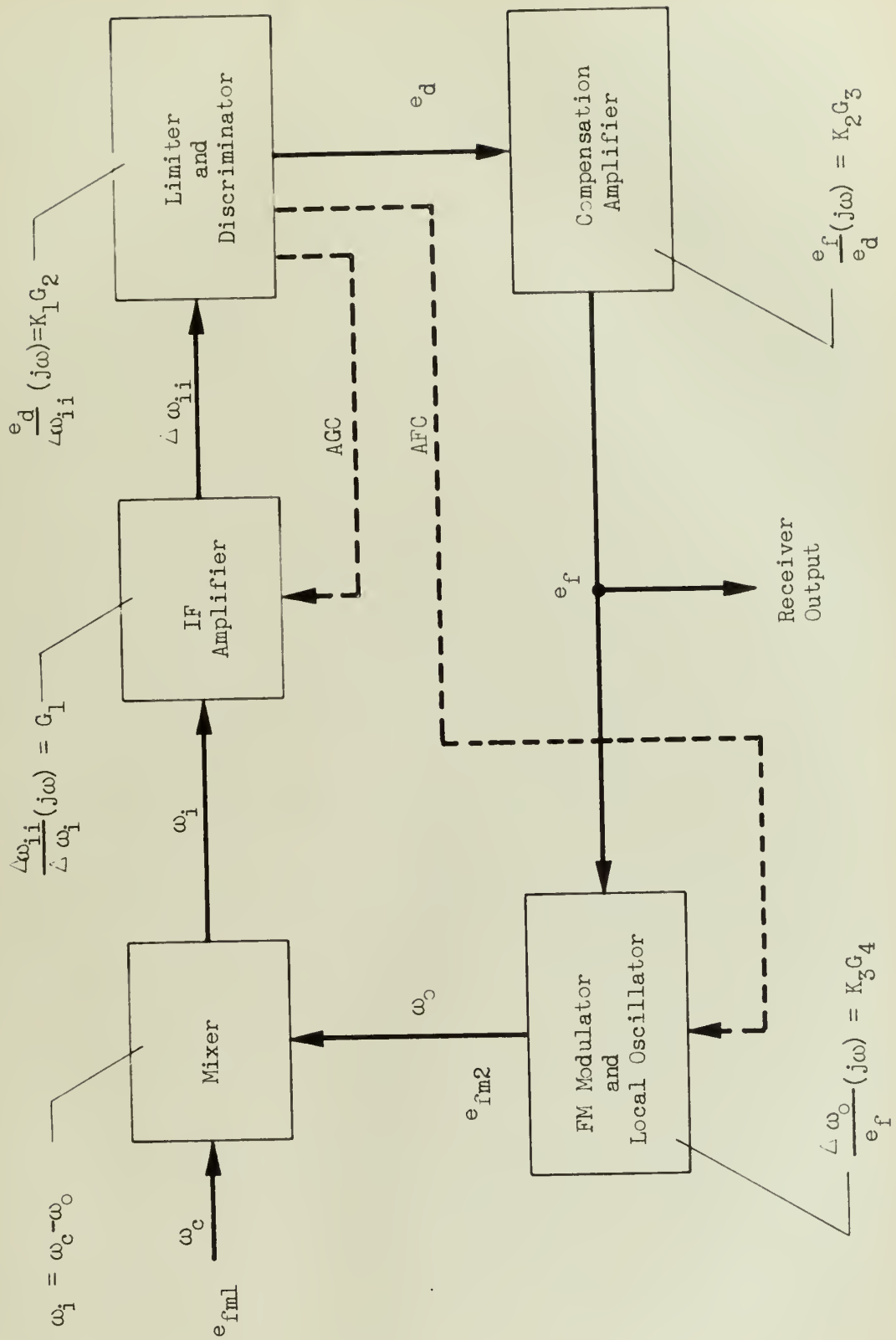
Considering the mixer as an ideal multiplier and noting that the output of the mixer is tuned to the difference frequency

$$e_o = E_1 E_2 \sin (\omega_c t + m_{f_1} \sin \omega_m t) \sin (\omega_o t + m_{f_2} \sin [\omega_m t + \theta]) \quad (4-3)$$

Expanding by $\sin x \sin y = 1/2 \cos (x - y) - 1/2 \cos (x + y)$ gives

$$e_o = E_1 E_2 \left\{ 1/2 \cos [(\omega_c t + m_{f_1} \sin \omega_m t) - (\omega_o t + m_{f_2} \sin (\omega_m t + \theta))] \right. \\ \left. - 1/2 \cos [(\omega_c t + m_{f_1} \sin \omega_m t) + (\omega_o t + m_{f_2} \sin (\omega_m t + \theta))] \right\} \quad (4-4)$$

$$e_o = \frac{E_1 E_2}{2} \left\{ \cos [(\omega_c - \omega_o)t + m_{f_1} \sin \omega_m t - m_{f_2} \sin (\omega_m t + \theta)] \right. \\ \left. - \cos [(\omega_c + \omega_o)t + m_{f_1} \sin \omega_m t + m_{f_2} \sin (\omega_m t + \theta)] \right\} \quad (4-5)$$



FM MODULATION FEEDBACK LOOP

Figure 2

The second term may be neglected since it is outside the pass band of the tuned circuit at the output of the mixer.

Expanding by $\cos (x + y) = \cos x \cos y - \sin x \sin y$ gives

$$e_o = \frac{E_1 E_2}{2} \left\{ \cos (\omega_c - \omega_o)t \cos [m_{f_1} \sin \omega_m t - m_{f_2} \sin (\omega_m t + \theta)] \right. \\ \left. - \sin (\omega_c - \omega_o)t \sin [m_{f_1} \sin \omega_m t - m_{f_2} \sin (\omega_m t + \theta)] \right\} \quad (4-6)$$

By placing an RC network at the output of the compensation amplifier the phase θ is made equal to zero and is stabilized by the high loop gain of the system. Thus

$$e_o = \frac{E_1 E_2}{2} \left\{ \cos (\omega_c - \omega_o)t \cos [(m_{f_1} - m_{f_2}) \sin \omega_m t] \right. \\ \left. - \sin (\omega_c - \omega_o)t \sin [(m_{f_1} - m_{f_2}) \sin \omega_m t] \right\} \quad (4-7)$$

Expanding by $\cos x \cos y = 1/2 \cos (x + y) + 1/2 \cos (x - y)$ gives

$$e_o = \frac{E_1 E_2}{4} \left\{ \cos [(\omega_c - \omega_o)t + (m_{f_1} - m_{f_2}) \sin \omega_m t] \right. \\ + \cos [(\omega_c - \omega_o)t - (m_{f_1} - m_{f_2}) \sin \omega_m t] \\ - \cos [(\omega_c - \omega_o)t + (m_{f_1} - m_{f_2}) \sin \omega_m t] \\ \left. + \cos [(\omega_c - \omega_o)t - (m_{f_1} - m_{f_2}) \sin \omega_m t] \right\} \quad (4-8)$$

$$e_o = \frac{E_1 E_2}{2} \cos [(\omega_c - \omega_o)t + (m_{f_1} - m_{f_2}) \sin \omega_m t] \quad (4-9)$$

This may be recognized as an FM wave of carrier frequency $(\omega_c - \omega_o)$ and modulation index of $(m_{f_1} - m_{f_2})$. Since $m_{f_2} \approx 1$ by design, the effective modulation index is seen to be on the order of .02. The above wave may also be written in Bessel function form as

$$\begin{aligned}
e_o = \frac{E_1 E_2}{2} & \left\{ J_0(m_{f_1} - m_{f_2}) [\cos(\omega_c - \omega_o)t] \right. \\
& + J_1(m_{f_1} - m_{f_2}) [\cos(\omega_c - \omega_o + \omega_m)t - \cos(\omega_c - \omega_o - \omega_m)t] \\
& + J_2(m_{f_1} - m_{f_2}) [\cos(\omega_c - \omega_o + 2\omega_m)t + \cos(\omega_c - \omega_o - 2\omega_m)t] \\
& + \dots \left. \right\} \quad (4-10)
\end{aligned}$$

To show the phase stability of the loop refer to the transfer functions indicated in Figure 2.

$$m_f = \frac{\Delta\omega}{\omega_m} \quad \text{where } \Delta\omega = \text{the maximum deviation of the carrier frequency} \quad (4-11)$$

$$\Delta\omega_c = m_{f_1} \omega_m \quad (4-12)$$

$$\Delta\omega_o = m_{f_2} \omega_m \quad (4-13)$$

$$\Delta\omega_i = (m_{f_1} - m_{f_2}) \omega_m \quad (4-14)$$

Assuming the discriminator to be linear over the region of operation, the discriminator output = $K_1 G_1 G_2 \Delta\omega_i(j\omega)$ (4-15)

$$\Delta\omega_i(j\omega) = \Delta\omega_c(j\omega) - \Delta\omega_o(j\omega) \quad (4-16)$$

$$\Delta\omega_o(j\omega) = \Delta\omega_i(j\omega) KG \quad (4-17)$$

$$\text{where } KG = K_1 K_2 K_3 G_1 G_2 G_3 G_4$$

$$\Delta\omega_i(j\omega) = \Delta\omega_c(j\omega) - \Delta\omega_i KG(j\omega) \quad (4-18)$$

$$\frac{\Delta\omega_i}{\Delta\omega_c}(j\omega) = \frac{1}{1 + KG(j\omega)} \quad (4-19)$$

$$e_f(j\omega) = \Delta\omega_i K_1 K_2 G_1 G_2 G_3(j\omega) \quad (4-20)$$

$$\frac{e_f}{\Delta\omega_c}(j\omega) = \frac{K_1 K_2 G_1 G_2 G_3(j\omega)}{1 + KG(j\omega)} \quad (4-21)$$

which is the closed loop transfer function for the receiver.

If $KG(j\omega) \gg 1$

$$\frac{e_f}{\Delta\omega_c}(j\omega) \approx \frac{1}{K_3 G_4} \quad (4-22)$$

Thus, the phase shift of the receiver output becomes independent of changes in the phase shift of the IF amplifier, limiter, discriminator and compensation amplifier.

The transfer function for the local oscillator is

$$\frac{\Delta\omega_o}{\Delta\omega_c} = \frac{KG(j\omega)}{1 + KG(j\omega)} \quad (4-23)$$

which for $KG \gg 1$ is approximately unity thus stabilizing the modulation indices.

B. AM Modulation Feedback Loop Analysis. (Refer to Figure 3).

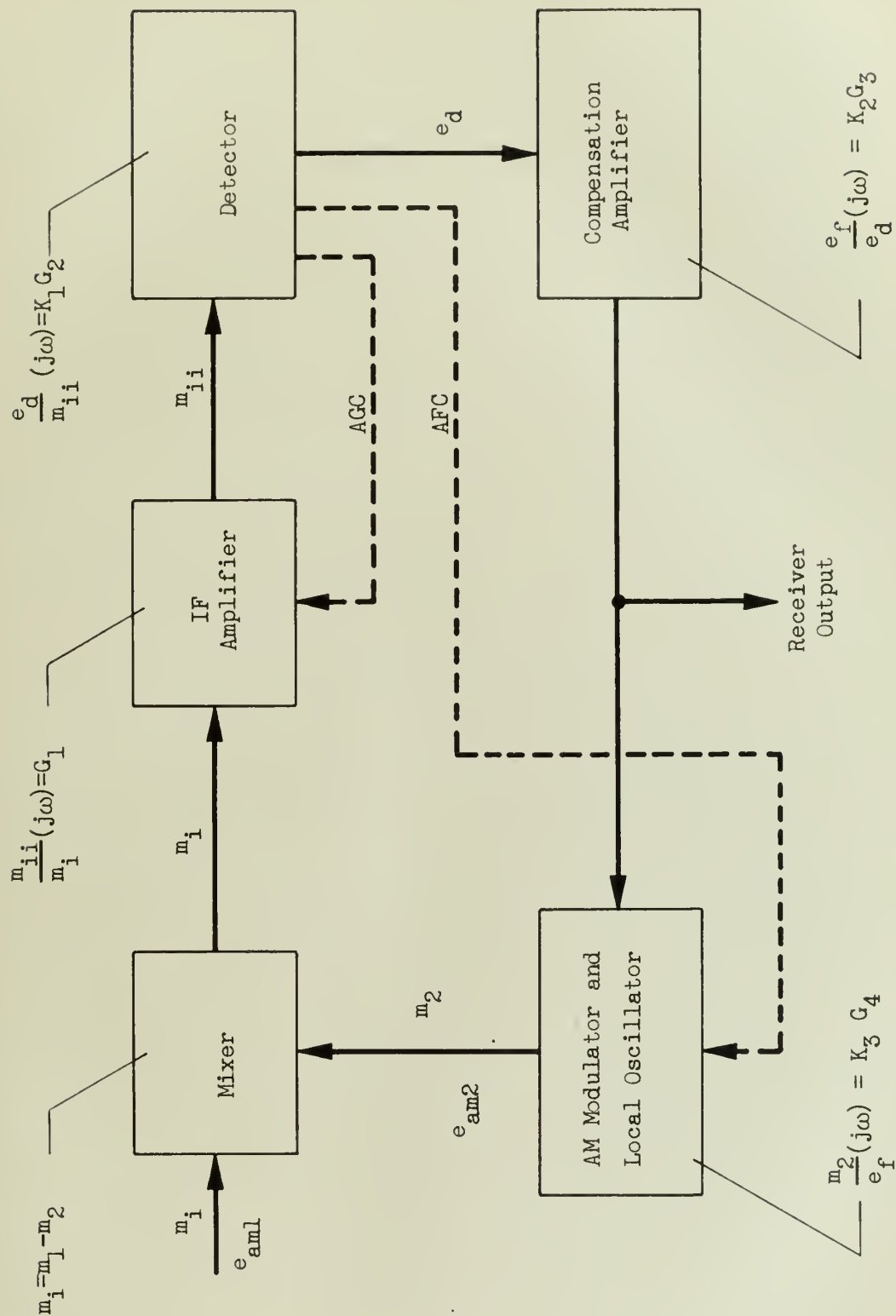
$$e_{am_1} = E_1 \sin \omega_c t [1 + m_1 \cos \omega_m t] \quad (4-24)$$

$$e_{am_2} = E_2 \sin \omega_o t [1 - m_2 \cos (\omega_m t + \theta)] \quad (4-25)$$

By placing RC network at the output of the compensation amplifier, the phase, θ , is designed to be zero. Considering the mixer as an ideal multiplier and noting that the output of the mixer is tuned to the difference frequency

$$e_o = E_1 E_2 \sin \omega_c t \sin \omega_o t [1 + (m_1 - m_2) \cos \omega_m t - m_1 m_2 \cos^2 \omega_m t] \quad (4-26)$$

$$= E_1 E_2 \left\{ \frac{1}{2} \cos (\omega_c - \omega_o) t - \frac{1}{2} \cos (\omega_c + \omega_o) t \right\} \\ \left\{ 1 + (m_1 - m_2) \cos \omega_m t - \frac{m_1^2 m_2}{2} - \frac{m_1 m_2}{2} \cos 2\omega_m t \right\} \quad (4-27)$$



AM MODULATION FEEDBACK LOOP

Figure 3



$$\begin{aligned}
&= E_1 E_2 \left\{ \frac{1}{2} \cos (\omega_c - \omega_o) t - \frac{1}{2} \cos (\omega_c + \omega_o) t + \frac{(m_1 - m_2)}{4} \right. \\
&\quad \left[\cos (\omega_c - \omega_o + \omega_m) t + \cos (\omega_c - \omega_o - \omega_m) t \right] - \frac{(m_1 - m_2)}{4} \\
&\quad \left[\cos (\omega_c + \omega_o + \omega_m) t + \cos (\omega_c + \omega_o - \omega_m) t \right] - \frac{m_1 m_2}{8} \\
&\quad \left[\cos (\omega_c - \omega_o + 2\omega_m) t + \cos (\omega_c - \omega_o - 2\omega_m) t \right] + \frac{m_1 m_2}{4} \\
&\quad \left. \cos (\omega_c + \omega_o) t + \frac{m_1 m_2}{8} \left[\cos (\omega_c + \omega_o + 2\omega_m) t + \cos (\omega_c + \omega_o - 2\omega_m) t \right] \right\}
\end{aligned}
\tag{4-28}$$

Neglecting the terms that are not passed by the tuned circuit at the output of the mixer

$$\begin{aligned}
e_o &= E_1 E_2 \left\{ \left(1 - \frac{m_1 m_2}{2} \right) \cos (\omega_c - \omega_o) t + \frac{(m_1 - m_2)}{2} \right. \\
&\quad \left. \left[\cos (\omega_c - \omega_o + \omega_m) t + \cos (\omega_c - \omega_o - \omega_m) t \right] \right\}
\end{aligned}
\tag{4-29}$$

This may be recognized as an AM wave of carrier frequency $(\omega_c - \omega_o)$, modulating frequency ω_m and modulation index $k (m_1 - m_2)$ where k has a value depending on the modulation index and normalizes the expression to the standard form of unity carrier coefficient. The effective modulation index is seen to be the small difference between m_1 and m_2 and is stabilized by the high loop gain to approximately .02.

To show the phase stability of the loop refer to the transfer functions indicated in Figure 3. The mixer acts as a multiplier and is tuned to the difference frequency between the input waves. The IF strip amplifies this difference. The detector output is proportional to the instantaneous amplitude input of the modulation component of the IF output signal provided linearity is assumed. In transfer function notation, noting that the instantaneous amplitude of modulation component is a function of frequency, this may be represented as follows:

$$m_i(j\omega) = m_1(j\omega) - m_2(j\omega) \quad (4-30)$$

$$m_2(j\omega) = m_i KG(j\omega) \quad (4-31)$$

$$\text{where } KG = K_1 K_2 K_3 G_1 G_2 G_3 G_4$$

$$m_i(j\omega) = m_1(j\omega) - m_i KG(j\omega) \quad (4-32)$$

$$\frac{m_i(j\omega)}{m_1} = \frac{1}{1 + KG(j\omega)} \quad (4-33)$$

$$e_f(j\omega) = m_i K_1 K_2 G_1 G_2 G_3(j\omega) \quad (4-34)$$

$$\frac{e_f(j\omega)}{m_1} = \frac{K_1 K_2 G_1 G_2 G_3(j\omega)}{1 + KG(j\omega)} \quad (4-35)$$

which is the closed loop transfer function for the receiver. If $KG(j\omega) \gg 1$, this reduces to $\frac{1}{K_3 G_4}$. Thus the phase shift of the receiver output becomes

independent of the phase shifts in the IF strip, detector and compensation amplifier shown in Figure 3. The transfer function of the local oscillator is

$$\frac{m_2(j\omega)}{m_1} = \frac{KG(j\omega)}{1 + KG(j\omega)} \quad (4-36)$$

which for $KG \gg 1$ is approximately unity thus stabilizing the modulation indices.

The results of the AM and FM analyses shown above proves that the fundamental standard for comparison of equivalent phase stability has been met. In addition the mixer outputs have been shown to be actually an AM or FM wave. The importance of the loop operation will be more fully appreciated later when the basic differences between the AM and FM systems will be shown to depend on the nature of the operation of the loop on the respective waves.

5. Second Order Modulation Effects.

The products formed in the mixing action of the modulation feedback loops can quite conceivably contribute second or higher order sidebands which will cause phase distortion in the systems or even first order sidebands which will contribute directly to the distortion of the modulation. A comparison of these effects for the two systems is therefore quite important.

Equation (4-10) shows that the output of the mixer for the FM case is

$$\begin{aligned}
 e_o = \frac{E_1 E_2}{2} & \left\{ J_0(m_{f_1} - m_{f_2}) [\cos(\omega_c - \omega_o)t] \right. \\
 & + J_1(m_{f_1} - m_{f_2}) [\cos(\omega_c - \omega_o + \omega_m)t - \cos(\omega_c - \omega_o - \omega_m)t] \\
 & + J_2(m_{f_1} - m_{f_2}) [\cos(\omega_c - \omega_o + 2\omega_m)t + \cos(\omega_c - \omega_o - 2\omega_m)t] \\
 & + \dots \dots \dots \left. \right\} \quad (5-1)
 \end{aligned}$$

For the DME system it is known that $m_{f_1} - m_{f_2}$ is approximately .02. From the table below it is quite obvious that for m_f up to .2 that the j_2 term will be at its maximum value only approximately .04 of the j_1 term and therefore there is no distortion present due to modulation products in the FM case. If the above conclusion is true then the expanded products of the two FM waves in terms of Bessel functions should be identical to equation (5-1). The mathematics is quite long and is contained in Appendix A. The result is identical with equation (5-1) and conclusively proves that there is no distortion due to the mixing action in the FM case.

m_f	.02	.2
j_0	.99990	.99902
j_1	.01000	.09950
j_2	.00005	.00498
j_3	.00000	.00016

An examination of equation (4-28) will show that for the AM case the output of the mixer is

$$e_{oAM} = \frac{E_1 E_2}{2} \left\{ \left(1 - \frac{m_1 m_2}{2} \right) \cos (\omega_c - \omega_o) t + \frac{(m_1 - m_2)}{2} \right. \\ \left[\cos (\omega_c - \omega_o + \omega_m) t + \cos (\omega_c - \omega_o - \omega_m) t \right] \\ \left. + \frac{m_1 m_2}{4} \left[\cos (\omega_c - \omega_o + 2\omega_m) t + \cos (\omega_c - \omega_o - 2\omega_m) t \right] \right\} \quad (5-2)$$

The third term of the above expression represents second harmonic distortion and is caused by the product of the two AM waves in the mixer. The magnitude of this term is comparable with or greater than the first order sidebands depending on the value of m . Ideally, one would desire to reject this component by the tuned circuits of the mixer output and IF stages. This would require high Q , narrow band, tuned circuits. The bandwidth, however, is required to be sufficiently wide to pass all five of the modulating frequencies and therefore this second harmonic term cannot be rejected. Taking the worse case, $m = 1$, the phase shift and amplitude response versus frequency may be obtained from equation (3-8), page 42 of Terman [24],

$$\theta = \tan^{-1} Q \frac{\delta (2 + \delta)}{(1 + \delta)^2} \quad (5-3)$$

$$\left| \frac{A_f}{A_o} \right| = \left| \frac{1}{1 + \delta + j Q \delta \frac{(2 + \delta)}{(1 + \delta)}} \right| \quad (5-4)$$

$$\text{where } \delta = \frac{f}{f_o} - 1$$

so that at the input to the detector the relative phase relationships caused by the phase shift through the IF amplifiers are as shown in Figure 4.

FM was shown to have no distortion due to the mixer action while it is quite obvious that second harmonic distortion will have three very serious effects in AM: (1) at the signal-to-noise threshold level the first order sidebands will be suppressed by the second order sidebands thus raising the effective threshold level; (2) the detector output will be a complex wave resulting from both pairs of sidebands and although the original modulation is recoverable by means of the tuned circuits at the output of the compensation amplifier in Figure 3, the phase shift resulting will seriously deteriorate the system accuracy, and (3) for synchronous detection undersirable cross modulation terms will result at the modulating frequency ω_{m1} .

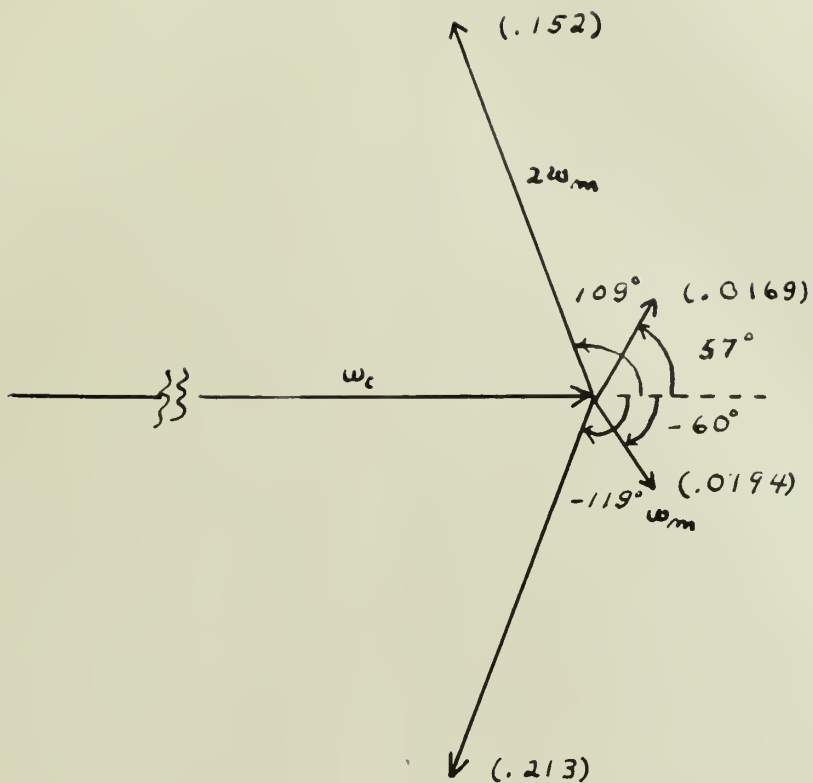


FIG. 4 PHASOR DIAGRAM AT DETECTOR INPUT

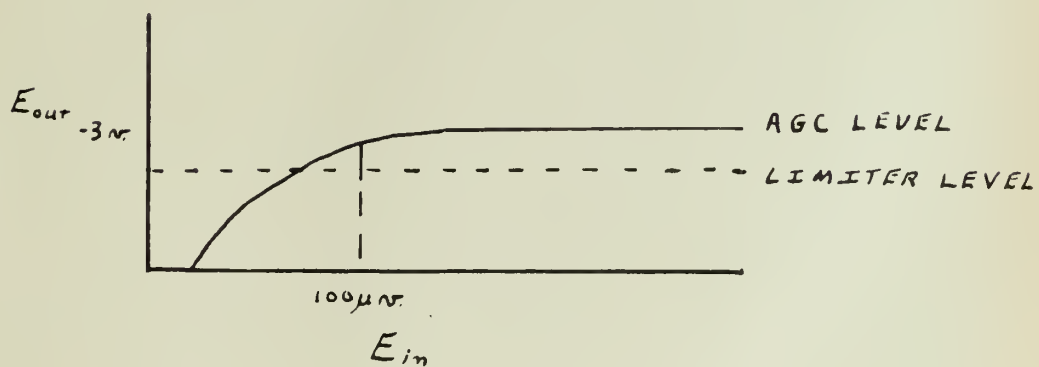


FIG. 5 AGC CHARACTERISTICS

6. Automatic Gain Control and Limiter Considerations.

The modulation indices of the FM and AM waves at the output of the modulators shown in Figures 2 and 3, respectively, are proportional to the peak amplitudes of the recovered modulation at the receiver output. For a constant index of modulation then the peak output modulation voltage should be constant. This leads naturally to a consideration of the AGC characteristics of the receiver.

Figure 5 is a typical AGC receiver characteristic for the DME and applies equally to both the AM and FM cases. Below a certain input level ($e_{in} = 100 \mu$ volts) the receiver output will become smaller and therefore the peak amplitude of the recovered modulation will become smaller. Taking the FM case with E_s the peak value of the output modulation voltage, then

$$m_{f_2} = k E_s \quad (6-1)$$

if it is assumed that the modulator action is linear.

Now
$$m_{f_1} = m_{f_1} - m_{f_2} \quad (6-2)$$

A reduction in E_s reduces m_{f_2} and hence increases the effective modulation index m_{f_1} . The results are similar for AM.

Another way of considering this problem is to note that the loop gain is a function of the input signal level; i.e., as the input signal level decreases, the loop gain decreases. Cubic has shown experimentally that a minimum of 20 db loop gain is required to maintain the phase stability requirements. This represents an effective modulation index of 0.1 and hence establishes the limiting input signal level required to maintain phase stability. Fortunately with the present system range and signal threshold level of 10 db input signal-to-noise ratio, both systems

operate essentially on the constant portion of AGC curves (above 100 microvolts input signal level) and therefore no problem is encountered in stabilizing the modulation indices. It should be noted, however, that the limiter in the FM case permits a 3 db smaller input signal level without deterioration of the effective modulation index and thus may be considered slightly better than the AM case.

Future systems, however, point to the detection of signals having -20 db input signal-to-noise ratio which do not provide sufficient loop gain to maintain phase stability. Jaffee and Rechtin [12] have described the inapplicability of "carrier only" AGC, which has the characteristic of maintaining E_s constant, to systems with this large a dynamic range of signal input levels. The AGC system employed by Cubic in the AME system utilizes the principles of heterodyning, narrow banding and amplification, with the characteristic of maintaining a constant power transfer, and succeeds in lowering the threshold level for AGC action to 1 to 5 microvolts input voltage vice the 100 microvolts in the present system. With the constant power transfer characteristic then as the (S/N) ratio input decreases more gain is required in the amplifier to maintain constant power and this available gain sets the lower level of detectable input signal which will provide a constant peak value of output modulation voltage and hence a stable modulation index. Fortunately again this technique provides operation on the constant portion of the AGC curve for both AM and FM for the lowest input (S/N) ratio envisioned for use in future systems.



7. Power Comparison.

Sheridan's [22] initial investigation of the relative power requirements between an AM and FM system indicated that AM required less power than FM for a given range by almost an order of magnitude. The results of his derivation is as follows:

$$\frac{P_{FM}}{P_{AM}} = \frac{m^2}{(1 + \frac{m^2}{2}) m_f^2} \quad (7-1)$$

where m = AM modulation index

m_f = FM modulation index

which relates the amount of FM signal power required in excess of AM signal power at the receiver input in order to obtain the same amount of data signal power at the receiver output under the assumption that the receiver has fixed gain and a given receiver output voltage is required. The constraint imposed on this equation was that $m_f \leq .2$, and that m could vary from 0 to 1. The validity of these constraints must be carefully examined.

The assumption that the receiver has fixed gain implies that the receiver has the same characteristics with respect to both waves. This further implies that the modulation feedback loop is used for both systems. This author has shown in Section 3 that modulation feedback is necessary in AM to obtain the same phase stability as FM and therefore is common to both systems for equivalent accuracy. With this criteria then $m \approx m_f \approx .02$ at the IF input and these respective modulation indices are stabilized at this value by the high loop gain of the respective feedback loops. We may recall from Section 4 that the feedback loop operation places no restraint on m or m_f at the input to the receiver but generates the small effective modulation indices as the difference between the input and regenerated indices.



To compare the relative power requirements between the FM and AM systems presents a very complex problem due to the inherent differences in the two waves, the complex modulation, and the different action by the modulation feedback loop on the two waves. Considering only one modulation frequency however, one can deduce a very useful comparison between the two systems. From Section 4 the mixer outputs may be written

$$e_{oAM} = \frac{E_1 E_2}{2} \left\{ \left(1 - \frac{m_1 m_2}{2} \right) \cos (\omega_c - \omega_o) t + \frac{(m_1 - m_2)}{2} \right. \\ \left. [\cos (\omega_c - \omega_o + \omega_m) t + \cos (\omega_c - \omega_o - \omega_m) t] \right\} \quad (7-2)$$

$$e_{oFM} = \frac{E_1 E_2}{2} \left\{ J_0(m_{f1} - m_{f2}) \cos (\omega_c - \omega_o) t + J_1(m_{f1} - m_{f2}) \right. \\ \left. [\cos (\omega_c - \omega_o + \omega_m) t - \cos (\omega_c - \omega_o - \omega_m) t] \right\} \quad (7-3)$$

since $(m_{f1} - m_{f2}) \approx .02$, equation (7-3) may be also written

$$e_{oFM} = \frac{E_1 E_2}{2} \left\{ \cos (\omega_c - \omega_o) t + \frac{(m_{f1} - m_{f2})}{2} \right. \\ \left. [\cos (\omega_c - \omega_o + \omega_m) t - \cos (\omega_c - \omega_o - \omega_m) t] \right\} \quad (7-4)$$

Normalizing equation (7-2) to have a unity carrier coefficient it becomes

$$e_{oAM} = \left(1 - \frac{m_1 m_2}{2} \right) \frac{E_1 E_2}{2} \left\{ \cos (\omega_c - \omega_o) t + \frac{(m_1 - m_2)}{2} \frac{1}{\left(1 - \frac{m_1 m_2}{2} \right)} \right. \\ \left. [\cos (\omega_c - \omega_o + \omega_m) t + \cos (\omega_c - \omega_o - \omega_m) t] \right\} \quad (7-5)$$

The total power contained in each wave may now be written in the form

$$P = A^2 \left(1 + \frac{m^2}{2} \right)$$

It is now possible to make different comparisons between the two waves depending on the criteria selected for comparison. The criteria vary with the type of detection employed so that comparisons will be made for the



two specific cases of detection discussed. First, however, it should be noted that one-half of the total FM power and slightly more of the total AM power, due to the reduction of the carrier, is lost because the output of the mixer is tuned to the difference frequency components. It is assumed that E_2 is the same for both systems.

Case I - Conventional Detection

The prime consideration for conventional detection is what is the relative sideband power at the detector input for the same total transmitted power?

$$P_{\text{total AM}} = E_{1 \text{ AM}}^2 \left[1 + \frac{m^2}{2} \right] \quad (7-6)$$

$$P_{\text{total FM}} = E_{1 \text{ FM}}^2 \quad (7-7)$$

$$\text{if } P_{\text{t AM}} = P_{\text{t FM}} \quad (7-8)$$

then

$$E_{1 \text{ FM}}^2 = \left[1 + \frac{m^2}{2} \right] E_{1 \text{ AM}}^2 \quad (7-9)$$

Since the effective modulation index is the same for both AM and FM by the action of the modulation feedback loop and is a small value such that the FM wave may be expressed as an AM wave, then the relative sideband power in the two systems may be expressed,

$$\left. \frac{P_{\text{FM}}}{P_{\text{AM}}} \right]_{\text{SB}} = \frac{\frac{m^2}{2} E_{1 \text{ FM}}^2}{\frac{m^2}{2} E_{1 \text{ AM}}^2} \quad (7-10)$$

where m = the effective modulation index.

Substituting equation (7-9) into equation (7-10)

$$\left. \frac{P_{\text{FM}}}{P_{\text{AM}}} \right]_{\text{SB}} = \left[1 + \frac{m^2}{2} \right] \quad (7-11)$$

which relates the power in the FM sidebands in excess of the AM sidebands.

The table below shows the relative sideband power in FM in excess of AM with AM modulation index as a parameter.

m	$\left[\frac{P_{FM}}{P_{AM}} \right]_{SB}$	$\left[\frac{P_{FM}}{P_{AM}} \right]_{SB} \text{ (db)}$
1	1.5	1.8
.8	1.32	1.2
.5	1.125	.5
.3	1.045	.2

Case II - Synchronous Detection

Since the active filter is used to derive the multiplying voltage for the synchronous detector then the carrier level becomes important in determining the range of carrier phase lock and hence the range of operation of the detector. For the same total transmitted power for both systems the sideband power will be related exactly as in the previous case but the carrier power is related as follows:

$$P_{\text{carrier AM}} = \left(1 - \frac{m_1 m_2}{2}\right)^2 E_{1 \text{ AM}}^2 \quad (7-12)$$

$$P_{\text{carrier FM}} = E_{1 \text{ FM}}^2 = \left(1 + \frac{m_1^2}{2}\right) E_{1 \text{ AM}}^2 \quad (7-13)$$

$$\frac{P_{c \text{ FM}}}{P_{c \text{ AM}}} = \frac{\left(1 + \frac{m^2}{2}\right)}{\left(1 - \frac{m_1 m_2}{2}\right)} \quad (7-14)$$

The table below shows the relative carrier power between the two waves with AM modulation index as a parameter.



m	$\frac{P_{cFM}}{P_{cAM}}$	$\left[\frac{P_{cFM}}{P_{cAM}} \right]$ db
1	3	4.78
.8	1.95	2.9
.5	1.28	1.1
.3	1.09	.35

Thus for conventional detection FM sideband power is approximately 1.8 db greater than the most favorable AM case. In addition to this advantage, the synchronous detector for the FM wave maintains phase lock with 4.78 db greater carrier power and is thus capable of detecting signals better than AM by this factor.



8. Range Comparison.

The improvement in range with AM indicated by Sheridan [22] is not valid since his derivation contained the same constraints as his power comparison, which were shown in the previous section to be incorrect. It has been well established by Terman [24] and others that range is proportional to the square root of the transmitted power. Applying this principle to the results of Section 7, then FM range in excess of AM range may be expressed

$$\frac{r_{FM}}{r_{AM}} = \sqrt{\left[\frac{P_{FM}}{P_{AM}} \right]_{SB}} = \sqrt{\left(1 + \frac{m^2}{2} \right)} \quad (8-1)$$

The table below shows the results of the above equation with m as a parameter.

m	$\frac{r_{FM}}{r_{AM}}$
1	1.22
.8	1.15
.5	1.06
.3	1.02

In addition to the above advantage, the synchronous detector for the FM case is capable of maintaining phase lock at ranges in excess of AM by the following:

$$\left[\frac{r_{FM}}{r_{AM}} \right]_{\text{phase lock}} = \sqrt{\frac{\left(1 + \frac{m^2}{2} \right)}{\left(1 - \frac{m_1 m_2}{4} \right)}} \quad (8-2)$$

The results of the above equation are shown in the table below with m as a parameter.

m	$\left. \frac{r_{FM}}{r_{AM}} \right\} \text{phase lock}$
1	1.73
.8	1.4
.5	1.13
.3	1.04



9. Conventional Detection.

Conventional detection means detection with the present threshold limitation. For this purpose, the AM system utilizes a standard envelope detector. The operational DME is a narrow band FM system utilizing a Foster-Seely discriminator with limiter. Assuming that the FM and AM inputs are essentially the same, then which system gives the best performance? There is an abundance of information in the literature by Middleton [16], Blachman [3] and others on the basic merits of these two systems. Since we are only interested here in the comparison between the two systems and not the detailed mathematical equations, then only the general results of the above references will be presented.

For weak carrier the FM receiver, at best (no limiting), is less effective than the AM receiver by 3 db while for strong carrier and (1) no limiting, the FM receiver approaches the performance of the AM receiver with a linear detector and (2) with heavy limiting, the FM receiver approaches the performance of the AM receiver with a square law device. Intermediate degrees of limiting find the behavior of FM between these two extremes unless clipping is especially heavy and the carrier weak when the (S/N) ratio output is noticeably diminished. Thus, the AM receiver is superior to the narrow band FM receiver for reception of weak or even moderate signals in fluctuation noise but for strong signals, the two systems compare favorably.



10. Optimum Demodulation

The objective of any receiver is to operate on a disturbance and to make a decision as to whether the disturbance came from noise alone or signal plus noise. Having made this decision, the receiver might additionally be required to present the parameters of the signal if the decision is reached that the disturbance came from signal plus noise. The DME system requires both the decision and the recovery of the signal if present.

With the above objective for a receiver, what then is the logical design approach to attain this objective? Birdsall, Peterson and Fox [2] have pointed out in their paper that the decision function basically stems from a likelihood criteria. This likelihood criteria is based on the statistical parameters of signal plus noise and noise alone.

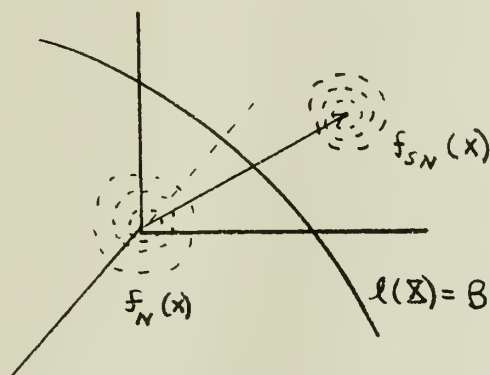


Fig. 6a Probability Distribution of Signal Plus Noise and Noise Alone

If the signal is known, then Figure 6a represents the three-dimensional probability distributions of noise, $f_N(x)$, and signal plus noise, $f_{SN}(x)$. (Three-dimensional surface chosen for convenience of illustration instead of actual N dimensions). For this case, $f_{SN}(x) = f(x - x_S)$. For a chosen hyper surface such that

$$l(x) = \frac{f_{SN}(x)}{f_N(x)} = \beta = \text{likelihood ratio} \quad (10-1)$$



then the observer will record

$$S + N \quad \text{for } l(x) \geq \beta$$

$$N \quad \text{for } l(x) < \beta$$

This paper further shows that all other common criteria for optimum demodulation reduce to this basic theory.

The DME system, however, has unknown carrier phase. Birdsall, Peterson and Fox [2] show that for this case, the unknown phase angle θ has a uniform distribution since all carrier phases are equally likely. They further show that the likelihood ratio becomes

$$l(x) = \exp \left[-\frac{E}{N_0} \right] \int \exp \left[\frac{1}{N} \sum x_i s_i \right] dP_s(\theta) \quad (10-2)$$

since the signal energy $E(S)$ is the same for all values of carrier phase θ and where

$$dP_s(\theta) = \frac{1}{2\pi} d\theta \quad (10-3)$$

Carrying out the integration over θ

$$l(x) = \exp \left[-\frac{E}{N_0} \right] I_0 \left(\frac{r}{N} \right) \quad (10-4)$$

Thus it is seen that the form of the receiver is a cross correlator with the output distribution of likelihood ratio a form of Rayleigh distribution.

The signal recovery function may be obtained from the principle of a posteriori probability used by Woodward [27] in his investigation of receiver design. The decision function for this approach stems from a likelihood criteria also as pointed out by Birdsall, Peterson and Fox [2]. A posteriori probability is inverse probability. In general, this means that information about the cause of a disturbance may be determined from the knowledge of its effects. In statistical terms, this means that the a priori probability of a signal occurring is known. The receiver looks



at the disturbance and is given the a priori probability of the signal being present and then produces the probability that the signal was actually present. In this form, the receiver is required to produce for each disturbance every possible message and its appropriate probability. We desire, therefore, to specify a receiver which will present the signal which has the maximum a posteriori probability or which picks the message which has the greatest probability of occurrence. This is essentially the maximum likelihood method. A very simple example of this theory is discussed by Lee [14] which relates the improvement in cross-correlation over auto correlation from a priori knowledge of frequency.

For our particular case, the above may be more appropriately understood from the elements of probability theory. If x and y are two random variables, then the joint probability $p(x, y)$ that a value x and a value y will occur together is

$$p(x, y) = p(x) p(y/x) = p(y) p(x/y) \quad (10-5)$$

where

$p(x)$ = probability that a value x will occur

$p(y)$ = probability that a value y will occur

$p(x/y)$ = probability, given a value y , that a value x will occur

$p(y/x)$ = probability, given a value x , that a value y will occur.

Solving the above equation

$$p(x/y) = \frac{p(x) p(y/x)}{p(y)} \quad (10-6)$$

Suppose x and y are each continuous functions; then the probability that y lies in an interval $(y + dy)$ is $p(y) dy$ and is now a continuous probability density and similarly the probability that x lies in an interval $(x + dx)$ is $p(x)dx$. Substituting such probabilities into the above equation, the dx 's and dy 's cancel but the p 's are now continuous probability density functions.



In general for any type modulation let

<u>function</u>	<u>probability density function</u>
$x = a(t) = \text{modulating intelligence}$	$p(a)$
$y = e_r(t) = \text{received signal}$	$p(r)$
$e_m[t, a(t)] = \text{modulated wave}$	$p(m)$
$e_n(t) = \text{noise} = e_r(t) - e_m[t, a(t)]$	$p(n)$

then

$$p[a(t)/e_r(t)] = \frac{p[a(t)] p[e_r(t)/a(t)]}{p[e_r(t)]} \quad (10-7)$$

Using the simplified notation above this becomes

$$p[a/r] = \frac{p(a) p(r/a)}{p(r)} \quad (10-8)$$

For a given received message $e_r(t)$, $p(r)$ is a constant. Its value is such that

$$\int_{-\infty}^{\infty} p(r/a) da = 1$$

$$\therefore p(a/r) = k_1 p(a) p(r/a) \quad (10-9)$$

If $a(t)$ is given (known modulating intelligence) then

$$p(r = x/a) = p(n = x - m) \quad (10-10)$$

thus

$$p(a/r) = k_1 p(a) p(n) \quad (10-11)$$

As a specific case, let us consider the FM wave of low modulation index present in the DME system.

$$e_m = E \sin \omega_c t + m_f E \sin \omega_m t \cos \omega_c t \quad (10-12)$$

$$e_r = e_m + e_n \quad (10-13)$$

$$e_n = e_r - e_m \quad (10-14)$$



Let us assume normal distribution for $e_n(t)$ and $a(t)$, that the received message exists in an interval $-T$ to T , and each wave has a constant power spectral density over some interval W . Then

$$p(a) = e^{-\frac{1}{A_o} \int_{-T}^T a^2(t) dt} \quad \text{where } A_o = \frac{\sigma_a^2}{W} \quad (10-15)$$

$$p(n) = e^{-\frac{1}{N_o} \int_{-T}^T e_n^2(t) dt} \quad \text{where } N_o = \frac{\sigma_n^2}{W} \quad (10-16)$$

$$p(a/r) = k_1 e^{-\frac{1}{A_o} \int_{-T}^T a^2(t) dt - \frac{1}{N_o} \int_{-T}^T e_n^2(t) dt} \quad (10-17)$$

$$p(a/r) = k_1 e^{-\int_{-T}^T \left[\frac{a^2(t)}{A_o} + \frac{(e_r - e_m)^2}{N_o} \right] dt} \quad (10-18)$$

$$= k_1 \exp \left[-\int_{-T}^T \left[\frac{a^2(t)}{A_o} + \frac{e_r^2(t) - 2e_r(t)e_m(t) + e_m^2(t)}{N_o} \right] dt \right] \quad (10-19)$$

Now $\int_{-T}^T e_r^2(t) dt$ for a received signal is constant and may be absorbed in the general constant

$$p(a/r) = k_2 \exp \left[-\int_{-T}^T \left[\frac{a^2(t)}{A_o} + \frac{e_m^2(t) - 2e_r(t)e_m(t)}{N_o} \right] dt \right] \quad (10-20)$$

To maximize this expression with respect to $a(t)$ to obtain the maximum likelihood, we must differentiate the integrand with respect to $a(t)$ and set equal to zero. Substituting from (10-12) the integrand becomes

$$\begin{aligned} & \frac{a^2(t)}{A_o} + \frac{E^2 \sin^2 \omega_c t + a^2(t) \cos^2 \omega_c t + 2E \sin \omega_c t \cos \omega_c t a(t)}{N_o} \\ & - \frac{2e_r(t) E \sin \omega_c t + 2e_r(t) \cos \omega_c t a(t)}{N_o} \end{aligned} \quad (10-21)$$



Differentiating with respect to $a(t)$ and setting equal to zero

$$\frac{2a(t)}{A_o} + \frac{2a(t)\cos^2 \omega_c t}{N_o} + \frac{2E \sin \omega_c t \cos \omega_c t}{N_o} - \frac{2e_r(t) \cos \omega_c t}{N_o} = 0 \quad (10-22)$$

$$a(t) \left[\frac{1}{A_o} + \frac{\cos^2 \omega_c t}{N_o} \right] = \frac{e_r(t)}{N_o} \cos \omega_c t - \frac{2E \sin \omega_c t \cos \omega_c t}{N_o} \quad (10-23)$$

$$a(t) \left[\frac{1}{A_o} + \frac{1}{2N_o} + \frac{1}{2} \frac{\cos 2\omega_c t}{N_o} \right] = \frac{1}{N_o} \cos \omega_c t \left[e_r(t) - E \sin \omega_c t \right] \quad (10-24)$$

The $2\omega_c t$ term may be neglected because it is filtered out. Thus

$$a(t) \left[\frac{N_o}{A_o} + \frac{1}{2} \right] = \cos \omega_c t \left[e_r(t) - E \sin \omega_c t \right] \quad (10-25)$$

$$a(t) = \cos \omega_c t \left[e_r(t) - E \sin \omega_c t \right] \left[\frac{A_o}{\frac{N_o}{2} + N_o} \right] \quad (10-26)$$

This represents the operations on the received signal to obtain the original modulation. It is essentially seen to be a cross correlator.

Thus "likelihood criteria" and "a posteriori probability" lead to the same type receiver for optimum demodulation - a cross correlation device. Thomas [25] also derives the optimum detector for AM and FM of small modulation index to be a cross-correlator. For application to the DME, the optimum filter for this detector is a comb type filter centered at each of the modulation frequencies and with the narrowest bandwidth feasible for noise rejection. Cubic has such a comb filter of 3 cycle bandwidth for each modulation frequency in the present system. This is essentially achieved by an application of the heterodyning principle. The optimum detector then takes the form shown in Figure 6b.

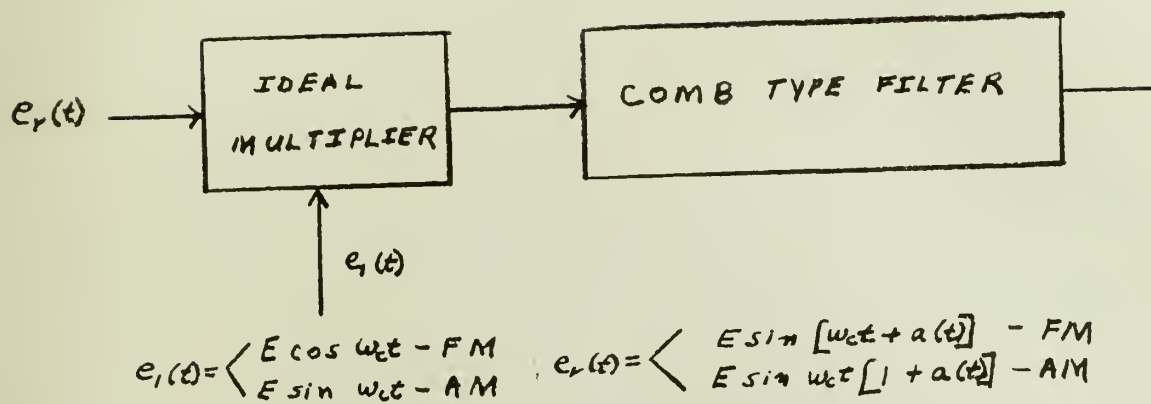


FIG. 6 b. OPTIMUM DETECTOR

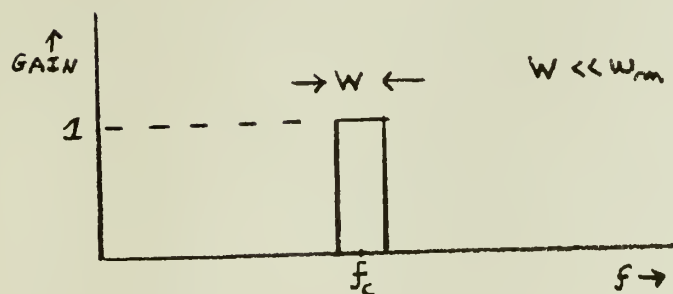


FIG. 7 IDEAL FILTER CHARACTERISTICS

The above results apply when the signal is known exactly; i.e., the multiplying voltage is in phase with the carrier of the received wave. For the DME this is not the case. The incoming carrier phase is unknown. To obtain the proper phase, it is necessary to filter out the carrier from the incoming wave and use it as the multiplying voltage. Since this is the actual case, in the DME, it is important to determine the effects of filtering.

Suppose we use an ideal band pass filter centered at f_c with gain 1 and bandwidth equal to W and $W \ll \omega_m$ as shown in Figure 7.

By the method of Rice [19] the filter output may be represented by

$$e_o = R(t) \cos [\omega_c t + \theta] \quad (10-27)$$

where R and θ are slowly varying functions of time compared to $\cos \omega_c t$ and with probability density distributions determined as follows:

$$E \cos \omega_c t = \text{incoming carrier wave} \quad (10-28)$$

$$E_c = \sum C_n \cos [(\omega_n - \omega_c)t - \phi_n] \quad (10-29)$$

$$E_s = \sum C_n \sin [(\omega_n - \omega_c)t - \phi_n] \quad (10-30)$$

$$E_n = E_c \cos \omega_m t - E_s \sin \omega_m t \quad (10-31)$$

$$\overline{E_n^2} = \overline{E_c^2} = \overline{E_s^2} = \phi_{(0)} = 2\sigma_t^2 \quad (10-32)$$

$$\text{the envelope} = R^2 = (E + E_c)^2 + E_s^2 \quad (10-33)$$

$$\text{let } x = E + E_c$$

$$y = E_s$$

$$\omega_1(x) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{(x-E)^2}{2\sigma_t^2}} \quad (10-34)$$

$$\omega_1(y) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{y^2}{2\sigma_t^2}} \quad (10-35)$$



$$\text{then } x = R \cos \theta$$

$$y = R \sin \theta$$

$$\omega_2(r, \theta) = \left\| J_{R\theta}^{xy} \right\| \omega_2(x, y) = \left\| \begin{array}{cc} \cos \theta & \sin \theta \\ -R \sin \theta & R \cos \theta \end{array} \right\| \omega_2(x, y) = R \omega_2(x, y) \quad (10-36)$$

$$\omega_2(x, y) = \frac{1}{2\pi\sigma_t^2} e^{-\frac{(x-E)^2}{2\sigma_t^2} - \frac{y^2}{2\sigma_t^2}} \quad (10-37)$$

$$\omega_2(R, \theta) = \frac{R}{2\pi\sigma_t^2} \exp \left[-\frac{(R^2 + E^2 - 2RE \cos \theta)}{2\sigma_t^2} \right] \quad (10-38)$$

$$\omega_1(R) = \int_0^{2\pi} \frac{R}{2\pi\sigma_t^2} e^{-\frac{(R^2 + E^2)}{2\sigma_t^2}} e^{\frac{RE \cos \theta}{\sigma_t^2}} d\theta \quad (10-39)$$

$$\omega_1(R) = \frac{R}{\sigma_t^2} \left[e^{-\left(\frac{R^2 + E^2}{2\sigma_t^2}\right)} \right] I_0 \left(\frac{RE}{\sigma_t^2} \right) \quad (10-40)$$

where I_0 is Bessel function of imaginary argument. It is shown in Appendix B that

$$\omega_1(\theta) = \frac{1}{\pi s} e^{-s^2} \frac{F}{\cos \theta} \quad (10-41)$$

where $s = \frac{E}{\sqrt{2}\sigma_t}$ so that $s^2 = \frac{\text{carrier}}{\text{noise}}$ power ratio

where noise refers to all waveforms which cause carrier phase ambiguity

and includes both $e_n(t)$ and contributions by $a(t)$. The distribution $\omega_1(R)$

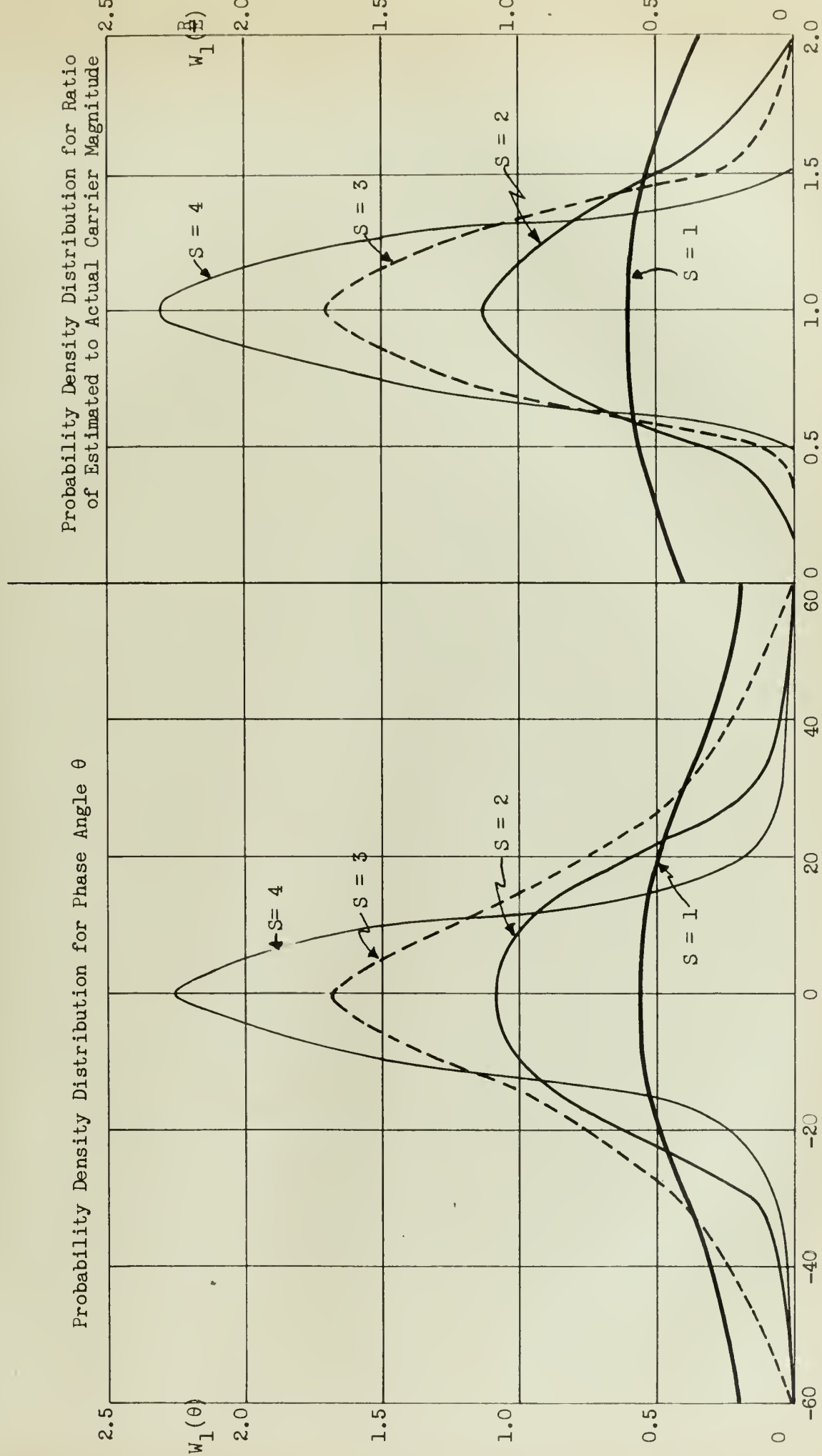
may be rewritten in normalized form from the relation $s = \frac{E}{\sqrt{2}\sigma_t}$

$$\omega_1\left(\frac{R}{E}\right) = 2s^2 \left(\frac{R}{E}\right) e^{-s^2 \left[\left(\frac{R}{E}\right)^2 + 1 \right]} I_0 \left(2s^2 \frac{R}{E} \right) \quad (10-42)$$

The distributions $\omega_1(\theta)$ and $\omega_1(\frac{R}{E})$ are plotted in Figure 8. It is quite obvious in both cases that the larger the value of s the more nearly are the distributions peaked around the desired values of $\theta = 0$ and $\frac{R}{E} = 1$ and therefore more closely does the system approach the theoretical optimum. Thus the R and θ distributions are a measure of the operation of the system. Practically the value of s is determined by the carrier filter width, the narrower the filter the greater the noise rejection and therefore the larger the value of s .

Theoretically we have shown that the form of the optimum receiver for both the AM and the FM case to be a cross correlator followed by a comb type filter when the signal is known exactly. To adapt this to the DME where the carrier phase is unknown then the multiplying voltage for cross correlation must be filtered from the incoming wave. It has further been shown that filtering will introduce errors of amplitude and phase in the multiplying voltage and that deviation from the correct value is proportional to s , the ratio of carrier to rms noise voltage. It has also been indicated that the narrower the filter bandwidth, the larger the value of s and hence the least error. The problem is now to devise methods to accomplish these results.





θ (degrees)

Figure 8

$(\frac{R}{E})$



11. Synchronous Detector

The synchronous detector proposed by Sheridan [22] utilizes the "active filter" of Jensen and McGeogh [13] to derive the multiplying voltage from the incoming wave and as shown in Figure 9 is the optimum detector derived in the previous section. Bandwidths of .01 cps centered up to 10 megacycles have been attained in operation with this filter with high attenuation of side frequencies. These filters may be also cascaded to provide additional attenuation of side frequencies if desired. Fundamentally this filter is a phase lock loop. The only difference in AM and FM is the 90° phase shift between the respective multiplying voltages.

Theoretically the threshold improvement results from the output (S/N) ratio being determined by the post detection bandwidth instead of the predetection bandwidth. It appears intuitively that with the effective modulation index $\approx .02$ as previously shown that the improvement in output (S/N) ratio should be the same for both FM and AM with the same inputs. The FM wave is analyzed in Appendix C and the results are shown below with the AM case derived by Sheridan.

$$\left(\frac{S}{N}\right)_{\text{out}}^{\text{FM}} = \left[\frac{m_f^2 \cos^2 \theta}{2\pi} \right] \left[\frac{A^2}{\beta N_o} \right] \quad (11-1)$$

$$\left(\frac{S}{N}\right)_{\text{out}}^{\text{AM}} = \left[\frac{m^2 \cos^2 \theta}{2\pi} \right] \left[\frac{A^2}{\beta N_o} \right] \quad (11-2)$$

Thus for the same modulation index the improvement in output (S/N) ratio is the same for both systems and is directly proportional to the post detection bandwidth. It should also be noted that the $\cos \theta$ term represents the phase deviation between the derived multiplying voltage and the incoming carrier wave.



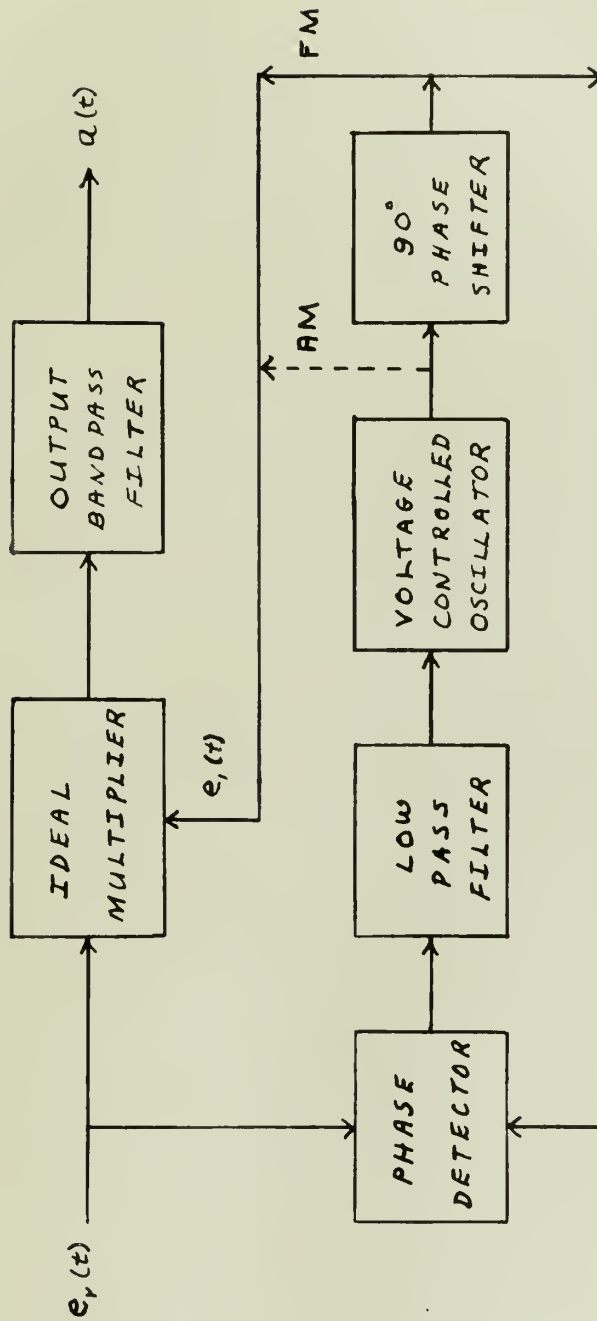


FIG. 9 SYNCHRONOUS DETECTOR



An analysis of the filter action is very complex due to non-linear operation of the phase lock loop. Jaffee and Rechtin [12] have originated a linear approximation to the loop and for input white gaussian noise have derived the following equation for the phase noise power:

$$\sigma_n^2 = \left(\frac{N}{2E^2 \Delta f} \right) 2f_n \quad (11-3)$$

where N = total noise power in output bandwidth of $\Delta f = N_o \Delta f$

E = amplitude of input wave

f_n = phase loop effective noise bandwidth

Now

$$s = \frac{E}{\sqrt{2}\sigma_n} \quad (11-4)$$

Substituting for σ_n from (11-3)

$$s = \frac{E}{\sqrt{2} \sqrt{\frac{N}{2E^2} \frac{2f_n}{\Delta f}}} = \frac{E^2}{\sqrt{2N_o}} \times \frac{1}{\sqrt{f_n}} = \frac{k}{\sqrt{f_n}}$$

This confirms the results of Section 10 which said that the smaller the loop bandwidth f_n , the larger the value of s , and hence more closely does the system approach the theoretical limit. It should be noted that there is a limit on the value of E which will maintain phase lock due to the sensitivity of the VCO. This may be considered as a large dispersion of R and θ about the desired values due to the small value of s . This problem will be discussed further in a later section. It should also be noted that the difference in phase between the input carrier and the multiplying voltage appears as a $\cos^2 \theta$ term in the output (S/N) ratios for both systems so that for large θ the deterioration in the output (S/N) ratio becomes very rapid.

12. Phase Detector as Correlator.

The system presently under development by Cubic as a correlation device is the phase detector employing phase modulation. Since essentially everything said prior to this about FM applies equally to PM then the comparison to be made here is AM versus PM and deserves consideration since it is the type of modulation used in the latest DME system. Skinner [23] has done a rather complete analysis on the phase detector which will not be repeated here except to show that the phase detector acts as a multiplier and therefore acts as a correlation device.

The mathematical operations performed in correlation techniques are multiplication and time averaging. These translate into spectrum shift and filtering respectively in the frequency domain. Heuristically we might note that if the phase detector of the synchronous detector acted as multiplier then a net reduction in the number of components and complexity of the receiver would result. Let us examine the phase detector under ideal conditions to see if it does perform the desired multiplying operation. The phase detector is shown in Figure 10.

For a PM wave

$$e_r = E \cos \omega_c t \quad (12-1)$$

$$e_{in} = E_s \sin [\omega_c t + \phi(t)] \quad (12-2)$$

The voltages applied between plates of T_1 and T_2 respectively are

$$e_1 = E_R \cos \omega_c t + E_s \sin [\omega_c t + \phi(t)] \quad (12-3)$$

$$e_2 = E_R \cos \omega_c t - E_s \sin [\omega_c t + \phi(t)] \quad (12-4)$$

Since ϕ is small

$$e_1 = (E_R + E_s \sin \phi) \cos \omega_c t + E_s \sin \omega_c t \quad (12-5)$$

$$e_2 = (E_R - E_s \sin \phi) \cos \omega_c t - E_s \sin \omega_c t \quad (12-6)$$



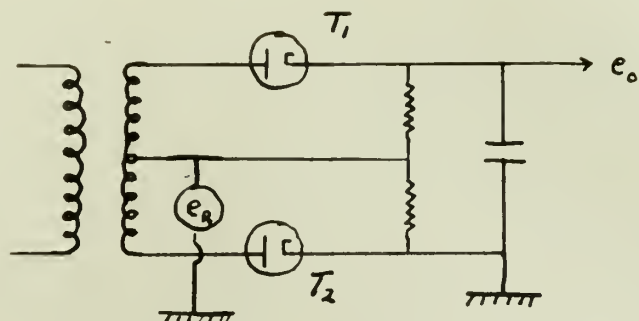


FIG. 10 PHASE DETECTOR

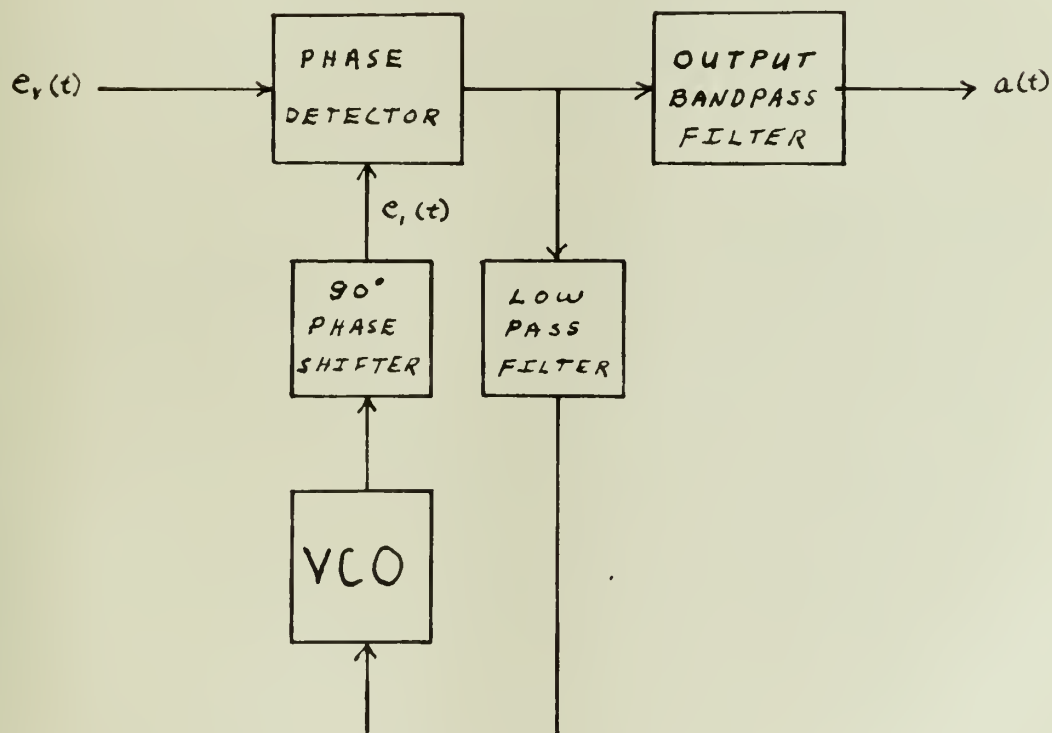


FIG. 11 PHASE DETECTOR CORRELATOR

With the diodes considered as linear rectifiers the envelope of the output voltage will be

$$e_o = \sqrt{(E_R + E_s \sin \phi)^2 + E_s^2} - \sqrt{(E_R - E_s \sin \phi)^2 + E_s^2} \quad (12-7)$$

which by expanding by the binominal series reduces to

$$e_o = 2E_s \sin \phi \left[1 - \frac{1}{2 \left(\frac{E_R}{E_s} \right)^2} \right] \quad (12-8)$$

Thus for large values of E_R compared to E_s which is the normal operating condition, then it is obvious that the intelligence shifted in spectrum is obtained as the output. Although the phase detector is not a multiplier in the usual sense, under the condition that the injected carrier is large compared to the received carrier, the desired translation of the intelligence is the same and hence the effect is that of cross correlation. The form of the optimum receiver thus becomes the phase detector correlator shown in Figure 11.

The phase detector correlator resembles the coherent detector of Middleton [17] with the exception that the resupplied carrier here depends on the phase lock loop for operation while Middleton used a separate generator with a variable phase in which phase lock could be obtained over any desired input level. Middleton's results with the coherent carrier shows that a theoretical upper limit of improvement of 38 db is possible with this system and obtained these results in the laboratory. Cubic's results have shown only a 20-25 db improvement due primarily to the phase lock problem when the input signal level is small. This problem will be discussed in a later section.

This system cannot be used for AM since the phase detector requires the injected carrier to be in phase quadrature with the incoming carrier to develop a control voltage and for this case the AM modulation is perturbed. This can easily be shown as follows:

$$e_{in} = \sin (\omega_c t + \theta) [1 + \phi(t)]$$

$$e_r = \cos \omega_c t$$

$$e_o = e_{in} e_r = \sin (\omega_c t + \theta) \cos \omega_c t + \phi(t) \sin (\omega_c t + \theta) \cos \omega_c t$$

$$\text{The low frequency output} = \frac{1}{2} \sin \theta + \frac{\phi(t)}{2} \sin \theta$$

which for small θ

$$e_{out} = \frac{\theta}{2} + \frac{\theta}{2} \phi(t)$$

The modulation is perturbed by the multiplying factor $(\frac{\theta}{2})$ which is a varying quantity. This result should also be recognized from the previous development which showed that the multiplying voltage for AM correlation should be in phase with incoming carrier.

13. Phase Lock Loop Considerations.

The phase lock loop forms an important part of both the synchronous detector and the phase detector correlation devices and an understanding of its limitations is extremely important. Jaffee and Rechtin [12] and Margolis [15] have both developed linear approximations to the loop from which the general operation may quite easily be understood. From Figure 12 the operation may be represented in simplest terms by the following:

$$e_i(t) = A_1 \sin(\omega_c t + \lambda_i) + N_1 f(t) \quad (13-1)$$

$$e_o(t) = \cos(\omega_c t + \lambda_{vco}) \quad (13-2)$$

$$e_{om} = A_1 \sin(\omega_c t + \lambda_i) \cos(\omega_c t + \lambda_{vco}) \quad (13-3)$$

where om = output of the multiplier

$$e_{om_{\text{low frequency}}} = A_1 (\lambda_i - \lambda_{vco}) \quad (13-4)$$

This dc voltage is passed by the RC filter, amplified and becomes the control voltage for the vco to maintain the phase lock. The limitation to be investigated is what is the minimum input signal that will provide sufficient voltage to maintain phase lock? The theoretical solution to this question has not been determined to the knowledge of the author although phase lock well below the noise level has been achieved in the laboratory. This limiting level is considered to be the reason that Cubic has not achieved the 38 db improvement predicted by Middleton for the coherent detector but that only 20 db has been realized.

The problem can easily be understood from the model of auto vs cross correlation. As the minimum input signal level is reached the output of the vco becomes $s_2(t) + n_2(t)$ while the input signal is $s_1(t) + n_1(t)$. For this case three noise terms are generated in the output of the multiplier instead of one as in cross correlation. This is shown in the below equations.

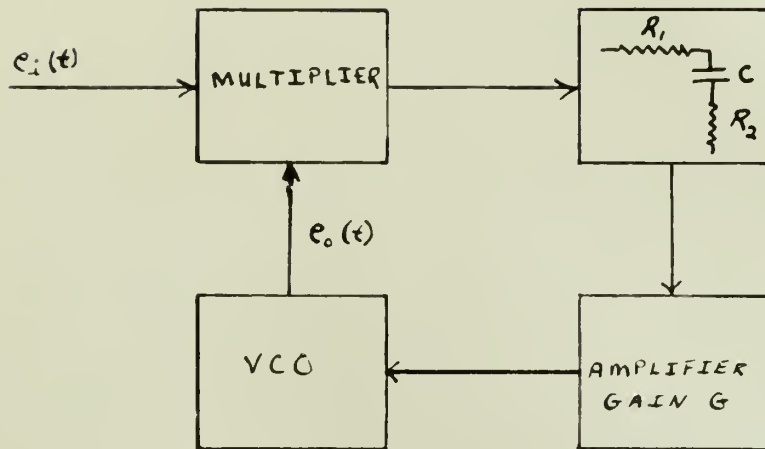


FIG. 12 PHASE LOCK LOOP

Auto correlation

$$\begin{aligned} \phi(\tau) &= \overline{[s_1(t) + n_1(t)] [s_2(t + \tau) + n_2(t + \tau)]} = \\ &\phi_{s_1 s_2}(\tau) + \phi_{s_1 n_2}(\tau) + \phi_{n_1 s_2}(\tau) + \phi_{n_1 n_2}(\tau) \end{aligned}$$

Cross correlation

$$\phi(\tau) = \overline{[s_1(t) + n_1(t)] [s_2(t + \tau)]} = \phi_{s_1 s_2}(\tau) + \phi_{n_1 s_2}(\tau)$$

The rapid deterioration of ability to maintain phase lock below this minimum level can be seen from the curves of Figure 13 developed by Lee [14] as a comparison of output (S/N) ratios between auto and cross correlation. Although this model does not determine the desired theoretical minimum level of operation of the loop it does indicate the nature of the problem and aids in an understanding of the circuit operation.

A more complex analysis based on the method of Margolis [15] permits calculation of the dc voltage produced by cross correlation between the input noise and the derived multiplying voltage. Referring to Figure 12 let

$$e_i(t) = A \sin(\omega_1 t + \lambda_1) + N_1 f(t)$$

$$e_o(t) = \cos(\omega_1 t + \lambda_{vco})$$

Then the low frequency output of the multiplier may be expressed

$$e_{1f} = \frac{A}{2} (\lambda_1 - \lambda_{vco}) + N_1 f(t) \cos(\omega_1 t + \lambda_{vco})$$

$$\text{Expanding in terms of } J = \frac{2N_1}{A_1} \text{ and } \lambda_{vco} - \lambda_1 = \lambda = \lambda_0 + J\lambda_1 + J\lambda_2^2 + \dots$$

$$e_{1f} = -\frac{A}{2} \lambda + N_1 f(t) \cos(\omega_1 t + \lambda_1 + \lambda)$$

$$= -\frac{A}{2} \lambda + N_1 f(t) \cos(\omega_1 t + \lambda_1) \cos \lambda + N_1 f(t) \sin(\omega_1 t + \lambda_1) \sin \lambda$$

$$\begin{aligned} = -\frac{A}{2} \lambda + N_1 f(t) \cos(\omega_1 t + \lambda_1) \underbrace{\left[1 - \frac{\lambda^2}{2} + \dots\right]}_{[1 - \lambda_0^2 + J^2 \lambda_1^2(t)]} + N_1 f(t) \sin(\omega_1 t + \lambda_1) \underbrace{\left[\lambda - \frac{\lambda^3}{6} + \dots\right]}_{[\lambda_0(t) + J\lambda_1(t) + J\lambda_2^2(t) + \dots]} \end{aligned}$$

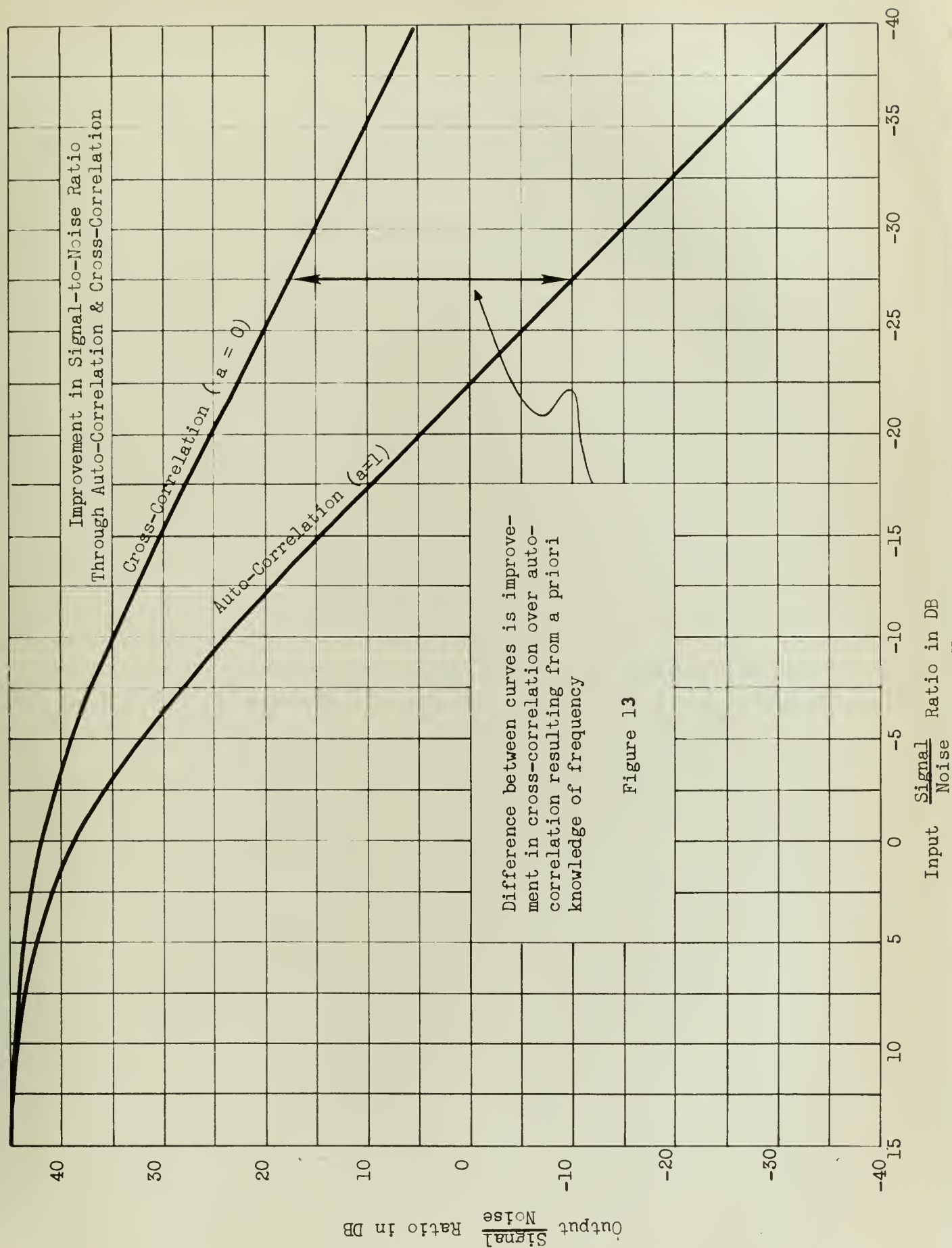


Figure 13

Regrouping terms for the second order approximation that contain only coefficients J

$$e_{lf} = -\frac{A}{2}\lambda + N_1 f(t) \cos(\omega_1 t + \lambda_1) + N_1 f(t) J\lambda_1 \sin(\omega_1 t + \lambda_1)$$

$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{dc control voltage} & \text{filtered out} & \text{dc term since } \lambda_1 \text{ is a filtered version of } f(t) \end{array}$

$$V_{dc} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left\{ J_{N1} \int_0^{\infty} h(\tau) f(t - \tau) \cos(\omega_1(t - \tau) + \lambda_1) d\tau \right\} f(t) \cos(\omega_1 t + \lambda_1) dt$$

If $f(t)$ has no periodic components

$$V_{dc} = \frac{JN_1}{2} \int_0^{\infty} \theta_{ff}(\tau) h(\tau) \cos \omega_1 \tau d\tau$$

where $h(\tau)$ is the inverse transform of $H(s)$

ϕ_{ff} is the auto correlation function of $f(t)$

For white noise through an ideal bandpass filter of bandwidth $2\Delta\omega$ centered at ω_1

$$\phi_{ff} = \sigma_f^2 \sin \frac{\Delta \omega T}{2} \cos \omega_1 T$$

$$V_{dc} = \frac{JN_1}{4} \sigma_f^2 \int_0^\infty \sin \frac{\Delta\omega\tau}{\Delta\omega_1\tau} (1 + \cos 2\omega_1\tau) h(\tau) d\tau$$

For good response

$$H(s) = \frac{\sqrt{2} \omega_N s + \omega_N^2}{s^2 + \sqrt{2} \omega_N s + \omega_N^2}$$

$$h(\tau) = \sqrt{2} \omega_n e^{-\frac{\sqrt{2}}{2} \omega_n \tau} \cos \frac{\sqrt{2}}{2} \omega_n \tau$$

$$V_{dc} = \frac{JN_1 \sigma_f^2}{4 \Delta \omega} \sqrt{2} \omega_N \int_0^\infty \sin \frac{\Delta \omega \tau}{\tau} e^{-\left(\frac{\sqrt{2}}{2}\right) \omega_N \tau} \cos \frac{\sqrt{2}}{2} \omega_N \tau d\tau$$

$$+ \frac{JN_1 \sigma_f^2}{4 \Delta \omega} \sqrt{2} \omega_N \int_0^\infty \sin \frac{\Delta \omega \tau}{\tau} e^{-\left(\frac{\sqrt{2}}{2}\right) \omega_N \tau} \cos \frac{\sqrt{2}}{2} \omega_N \tau \cos 2\omega_1 \tau d\tau$$

For $\Delta \omega$ and ω_N small compared to ω_1 then the second term may be neglected compared to the first, thus

$$V_{dc} = \frac{\sqrt{2}}{2} \left(\frac{\sigma_1^2}{A} \right) \left(\frac{\omega_N}{\Delta \omega} \right) \left[\frac{1}{2} \arctan \frac{\sqrt{2} \frac{\Delta \omega}{\omega_N}}{1 - \left(\frac{\Delta \omega}{\omega_N} \right)^2} \right]$$

which for $\Delta \omega \gg \omega_N$ i.e., noise bandwidth $\gg$ loop bandwidth

$$V_{dc} = \frac{\sqrt{2}}{2} \frac{\pi}{2} \left(\frac{\sigma_1^2}{A_1} \right) \left(\frac{\omega_N}{\Delta \omega} \right)$$

and for $\Delta \omega \ll \omega_N$ i.e., noise bandwidth $\ll$ loop bandwidth

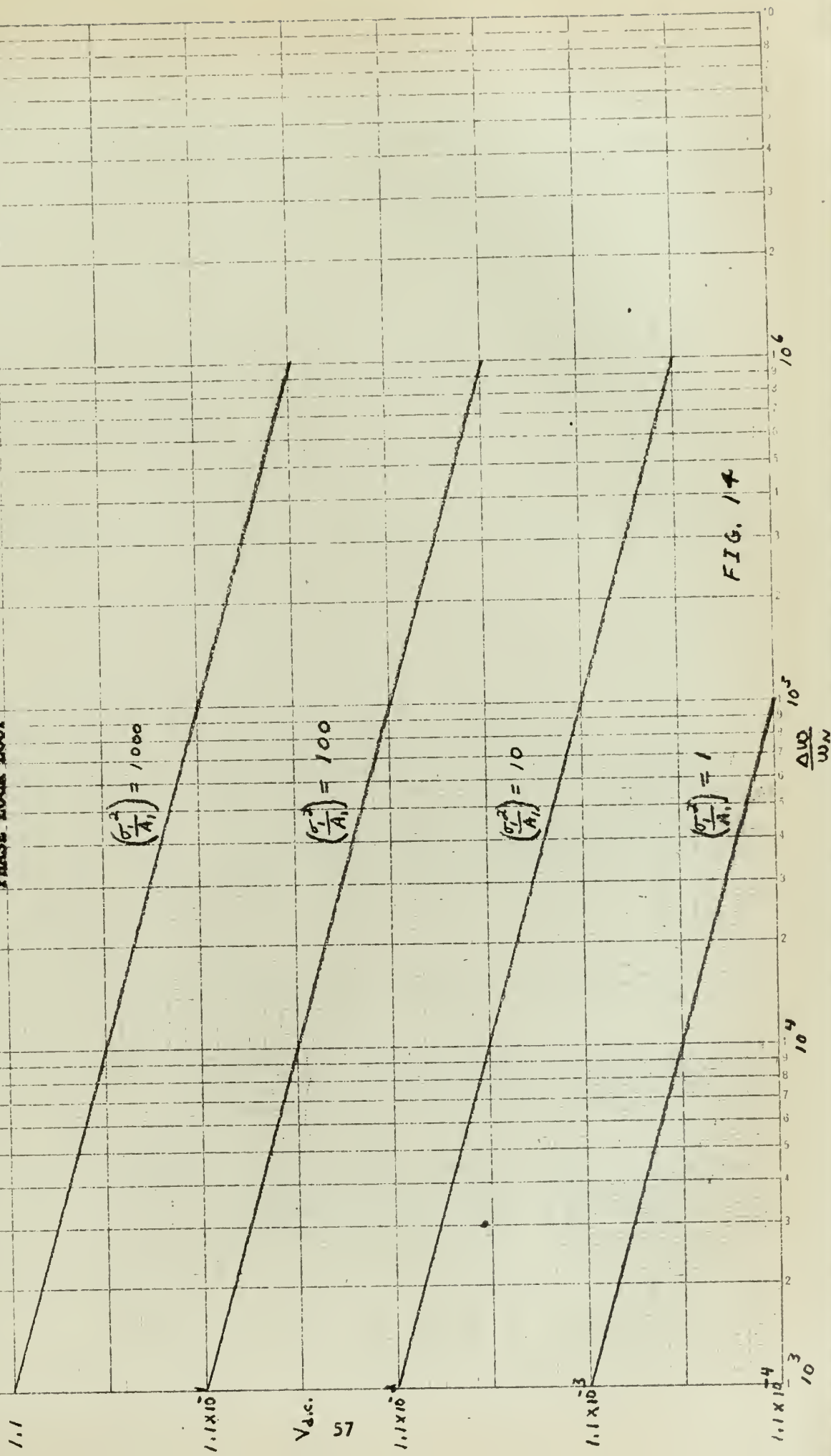
$$V_{dc} = \frac{1}{2} \frac{\sigma_1^2}{A_1}$$

A plot of V_{dc} versus the ratio of the noise to loop bandwidth $\left(\frac{\Delta \omega}{\omega_N} \right)$ for various values of the ratio of noise power to signal amplitude, $\left(\frac{\sigma_1^2}{A} \right)$,

is contained in Figure 14. Ranges of practical interest for the variables were chosen. The dc voltage may be converted to the frequency drift of the vco if the sensitivity of the vco is known. Although here again the theoretical minimum input signal level is not specifically determined, the curves do show the effective frequency drift of the vco as the input signal level decreases and the deterioration of the output (S/N) ratio may be determined for any specified input parameters with a small amount of computation.



D.C. VOLTAGE PRODUCED BY CROSS CORRELATION OF
INPUT NOISE WITH MULTIPLYING VOLTAGE OF THE
PHASE LOCK LOOP





14. Conclusions.

Frequency modulation has been shown to be superior to amplitude modulation for the DME under the required criteria of equal phase stability. Phase stability is attained through modulation feedback and in Section 4 it has been shown that the receiver loop transfer functions are identical for the two types of modulation. Based on the feedback loop operation, the following system characteristics were examined with the results indicated below:

	<u>FM</u>	<u>AM</u>
1. Second Order Modulation Effects (Section 5)	None	Large second harmonic distortion generated in the mixer
2. AGC (Section 6)	Same	
3. Power (Section 7)		
(a) With Conventional Detection	FM has up to 1.8 db more power in the sidebands at the input to the detector than AM.	
(b) With Synchronous Detection	FM has the above advantage in (a) plus up to 4.8 db more carrier power available at the detector input to maintain phase lock in the correlation detector.	
4. Range (Section 8)		
(a) With Conventional Detection	FM has the capability of up to 1.22 times the AM range.	
(b) With Synchronous Detection	FM has the above advantage in (a) plus up to 1.73 times the AM range in maintaining phase lock in the correlation device.	
5. Conventional Detection (Section 9)	AM has 3 db advantage for reception of weak or even moderate signals in fluctuation noise but for strong signals the systems are essentially the same.	

6. Correlation Detection
(Section 11)

Same improvement obtainable with both systems.

It is quite obvious that the FM superiority results from having no second order modulation effects and its extended phase lock capability with synchronous detection. For the remaining characteristics, the two systems are essentially the same. As pointed out earlier, the effective reduction in AM modulation index by using modulation feedback is the prime cause of AM system deterioration compared to FM. The necessity of using this feedback technique to obtain the required phase stability has been shown to be absolutely essential, however, in Section 3.

The phase detector has been shown to be a correlation device. It employs phase modulation and is similar to the coherent detector of Middleton [24]. Some problems in determining the input signal threshold level required to maintain phase lock in the loop have been discussed to obtain a better understanding of the loop performance. The theoretical threshold level, however, has not been determined and the maximum improvement possible with this system compared to the 38 db computed by Middleton [24] for the coherent detector has not been solved. Since the phase lock loop is used to derive the multiplying voltage for the synchronous detector, this problem additionally affects the absolute improvement possible with this technique. Essentially, the problem has been shown to result from the finite bandwidth required to derive the locking voltage. Curves have been developed in which the frequency drift of the vco of the phase lock loop has been shown to increase as the input signal level decreases. It is believed that this is a step towards the solution of the above problem.

15. Recommendations for Further Work.

The importance of the phase lock loop in the operation of both correlation detectors considered is self-evident. A study to determine the threshold of input signal level to maintain phase lock and a mathematical model to substantiate experimental results is considered highly desirable. The results of such a study will determine the theoretical improvement in output signal-to-noise ratio that is feasible with this technique compared to the 38 db improvement computed by Middleton [17] for the coherent detector.

Since the DME system utilizes only one-half of the total power available, then consideration of other modulation techniques appears appropriate. Single sideband, double sideband suppressed carrier, or even pulse techniques may prove feasible with large reduction in power requirements. In analyzing other modulation schemes, however, the technique employed to obtain system stability must be carefully investigated. It was this latter consideration that effectively made FM superior to AM.



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APPENDIX A

Proof, that the product of two FM waves in Bessel form is identical to equation (5-1)

$$e_{fm_1} = E_1 \left\{ J_0(mf_1) \sin \omega_c t \right. \\ + J_1(mf_1) [\sin(\omega_c + \omega_m)t - \sin(\omega_c - \omega_m)t] \\ + J_2(mf_1) [\sin(\omega_c + 2\omega_m)t + \sin(\omega_c - 2\omega_m)t] \\ \left. + \dots \right]$$

$$e_{fm_2} = E_2 \left\{ J_0(mf_2) \sin \omega_o t \right. \\ + J_1(mf_2) [\sin(\omega_o + \omega_m)t - \sin(\omega_o - \omega_m)t] \\ + J_2(mf_2) [\sin(\omega_o + 2\omega_m)t + \sin(\omega_o - 2\omega_m)t] \\ \left. + J_3(mf_2) \dots \right]$$

$$e_o = E_1 E_2 \left\{ J_0(mf_1) J_0(mf_2) \sin \omega_c t \sin \omega_o t \right. \\ + J_1(mf_1) J_0(mf_2) [\sin(\omega_c + \omega_m)t - \sin(\omega_c - \omega_m)t] \sin \omega_o t \\ + J_2(mf_1) J_0(mf_2) [\sin(\omega_c + 2\omega_m)t + \sin(\omega_c - 2\omega_m)t] \sin \omega_o t \\ + J_0(mf_1) J_1(mf_2) \sin \omega_c t [\sin(\omega_o + \omega_m)t - \sin(\omega_o - \omega_m)t] \\ + J_1(mf_1) J_1(mf_2) [\sin(\omega_c + \omega_m)t - \sin(\omega_c - \omega_m)t] \\ \quad [\sin(\omega_o + \omega_m)t - \sin(\omega_o - \omega_m)t] \\ + J_2(mf_1) J_1(mf_2) [\sin(\omega_c + 2\omega_m)t + \sin(\omega_c - 2\omega_m)t] \\ \quad [\sin(\omega_o + \omega_m)t - \sin(\omega_o - \omega_m)t] \\ + J_0(mf_1) J_2(mf_2) [\sin \omega_c t] [\sin(\omega_o + 2\omega_m)t + \sin(\omega_o - 2\omega_m)t] \\ + J_1(mf_1) J_2(mf_2) [\sin(\omega_c + \omega_m)t - \sin(\omega_c - \omega_m)t] \\ \quad [\sin(\omega_o + 2\omega_m)t + \sin(\omega_o - 2\omega_m)t] \\ \left. \right\}$$



$$\begin{aligned}
& + J_2(mf_1) J_2(mf_2) [\sin(\omega_c + 2\omega_m)t + \sin(\omega_c - 2\omega_m)t] \\
& \quad [\sin(\omega_o + 2\omega_m)t + \sin(\omega_o - 2\omega_m)t] \\
& + \dots \}
\end{aligned}$$

Now letting $J_o(mf_1) = J_{01}$ and similarly for the other terms and expanding

$$\begin{aligned}
e_o = E_1 E_2 \{ & J_{01} J_{02} \left[\frac{1}{2} \cos(\omega_c - \omega_o)t - \frac{1}{2} \cos(\omega_c + \omega_o)t \right] \\
& + J_{11} J_{02} \left[\frac{1}{2} \cos(\omega_c - \omega_o + \omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o + \omega_m)t \right. \\
& \quad \left. - \frac{1}{2} \cos(\omega_c - \omega_o - \omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o - \omega_m)t \right] \\
& + J_{21} J_{02} \left[\frac{1}{2} \cos(\omega_c - \omega_o + 2\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o + 2\omega_m)t \right. \\
& \quad \left. + \frac{1}{2} \cos(\omega_c - \omega_o - 2\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o - 2\omega_m)t \right] \\
& + J_{01} J_{12} \left[\frac{1}{2} \cos(\omega_c - \omega_o - \omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o + \omega_m)t \right. \\
& \quad \left. - \frac{1}{2} \cos(\omega_c - \omega_o + \omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o - \omega_m)t \right] \\
& + J_{11} J_{12} \left[\frac{1}{2} \cos(\omega_c - \omega_o)t - \frac{1}{2} \cos(\omega_c + \omega_o)t \right. \\
& \quad - \frac{1}{2} \cos(\omega_c - \omega_o + 2\omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o)t \\
& \quad - \frac{1}{2} \cos(\omega_c - \omega_o - 2\omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o)t \\
& \quad \left. + \frac{1}{2} \cos(\omega_c - \omega_o)t - \frac{1}{2} \cos(\omega_c + \omega_o - 2\omega_m)t \right] \\
& + J_{21} J_{12} \left[\frac{1}{2} \cos(\omega_c - \omega_o - \omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o + 3\omega_m)t \right. \\
& \quad - \frac{1}{2} \cos(\omega_c - \omega_o + 3\omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o + \omega_m)t \\
& \quad + \frac{1}{2} \cos(\omega_c - \omega_o - 3\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o - \omega_m)t \\
& \quad \left. - \frac{1}{2} \cos(\omega_c - \omega_o - \omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o - 3\omega_m)t \right]
\end{aligned}$$



$$\begin{aligned}
& + J_{01} J_{22} \left[\frac{1}{2} \cos(\omega_c - \omega_o - 2\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o + 2\omega_m)t \right. \\
& \quad \left. + \frac{1}{2} \cos(\omega_c - \omega_o + 2\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o - 2\omega_m)t \right] \\
& + J_{11} J_{22} \left[\frac{1}{2} \cos(\omega_c - \omega_o - \omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o + 3\omega_m)t \right. \\
& \quad + \frac{1}{2} \cos(\omega_c - \omega_o + 3\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o - \omega_m)t \\
& \quad - \frac{1}{2} \cos(\omega_c - \omega_o - 3\omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o + \omega_m)t \\
& \quad \left. - \frac{1}{2} \cos(\omega_c - \omega_o + \omega_m)t + \frac{1}{2} \cos(\omega_c + \omega_o - 3\omega_m)t \right] \\
& + J_{21} J_{22} \left[\frac{1}{2} \cos(\omega_c - \omega_o)t - \frac{1}{2} \cos(\omega_c + \omega_o + 4\omega_m)t \right. \\
& \quad + \frac{1}{2} \cos(\omega_c - \omega_o + 4\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o)t \\
& \quad + \frac{1}{2} \cos(\omega_c - \omega_o - 4\omega_m)t - \frac{1}{2} \cos(\omega_c + \omega_o)t \\
& \quad \left. + \frac{1}{2} \cos(\omega_c - \omega_o)t - \frac{1}{2} \cos(\omega_c + \omega_o - 4\omega_m)t \right] \\
& + \dots \dots \dots \}
\end{aligned}$$

Regrouping in the form of equation (5-1) and considering the difference frequency components only due to the tuned circuit at the output of the mixer

$$\begin{aligned}
e_o = \frac{E_1 E_2}{2} & \left\{ [J_{01} J_{02} + 2J_{11} J_{12} + \dots] [\cos(\omega_c - \omega_o)t] \right. \\
& + [J_{11} J_{02} - J_{01} J_{12} + J_{21} J_{12} - J_{11} J_{22} + \dots \dots] \\
& \quad [\cos(\omega_c - \omega_o + \omega_m)t - \cos(\omega_c - \omega_o - \omega_m)t] \\
& + [J_{21} J_{02} - J_{11} J_{12} + J_{01} J_{22} - \dots \dots \dots] \\
& \quad [\cos(\omega_c - \omega_o + 2\omega_m)t + \cos(\omega_c - \omega_o - 2\omega_m)t] \\
& \left. + \dots \dots \dots \right\}
\end{aligned}$$



Now using formula 87 from the Table of Formulas in McLachlan [18]

$$J_n(u \pm v) = \sum_{r=-\infty}^{\infty} J_r(v) J_{n \mp r}(u) \quad \text{where } |v| < |u|$$

plus the relation $J_{-n}(Z) = (-1)^n J_n(Z)$

$$\begin{aligned} \text{then } J_1(mf_1 - mf_2) &= J_{02} J_{11} + J_{12} J_{21} + J_{22} J_{31} + \dots \\ &+ J_{-12} J_{01} + J_{-22} J_{-11} + \dots \end{aligned}$$

$$J_1(mf_1 - mf_2) = J_{02} J_{11} - J_{12} J_{01} + J_{12} J_{21} - J_{22} J_{11} + \dots$$

Similarly

$$J_0(mf_1 - mf_2) = J_{02} J_{01} + 2J_{11} J_{12} + 2J_{22} J_{21} + \dots$$

Thus the two waves are identical and there are no second order modulation effects in the FM system.



APPENDIX B

Derivation of the distribution of $\omega_1(\theta)$

$$\omega_1(\theta) = \int_0^\infty \frac{R}{2\pi\sigma_t^2} e^{-\frac{R^2+E^2}{2\sigma_t^2}} e^{\frac{RE \cos \theta}{\sigma_t^2}} dR$$

$$\text{let } K = 2\sigma_t^2$$

$$\omega_1(\theta) = \frac{e^{-\frac{E^2}{K}}}{\pi K} \int_0^\infty R e^{-\frac{(R-E \cos \theta)^2}{K} + \frac{E^2 \cos^2 \theta}{K}} dR$$

$$= \frac{e^{b^2} e^{-\frac{E^2}{K}}}{\pi} \left[\int_0^\infty \frac{(R-E \cos \theta)}{\sqrt{K}} e^{-\frac{(R-E \cos \theta)^2}{K}} \frac{dR}{\sqrt{K}} + \int_0^\infty \frac{E \cos \theta}{\sqrt{K}} e^{-\frac{(R-E \cos \theta)^2}{K}} \frac{dR}{\sqrt{K}} \right]$$

$$\text{where } b = \frac{E \cos \theta}{\sqrt{K}}$$

$$\text{let } x = \frac{R-E \cos \theta}{\sqrt{K}} \quad \begin{cases} R = 0 & R = \infty \\ x = \frac{-E \cos \theta}{\sqrt{K}} = -b & x = \infty \end{cases}$$

$$\omega_1(\theta) = \frac{e^{b^2} e^{-\frac{E^2}{K}}}{\pi} \left[\int_0^\infty x e^{-x^2} dx + \frac{E \cos \theta}{\sqrt{K}} \int_0^\infty e^{-x^2} dx \right]$$

$$= \frac{e^{b^2} e^{-\frac{E^2}{K}}}{\pi} \left[\frac{1}{2} + \frac{b}{2} \sqrt{\pi} - \int_0^b x e^{-x^2} dx + b \int_0^b e^{-x^2} dx \right]$$



$$\text{let } F = b e^{b^2} \left[\frac{1}{2} + \frac{b}{2} \sqrt{\pi} - \int_0^b x e^{-x^2} dx + b \int_0^b e^{-x^2} dx \right]$$

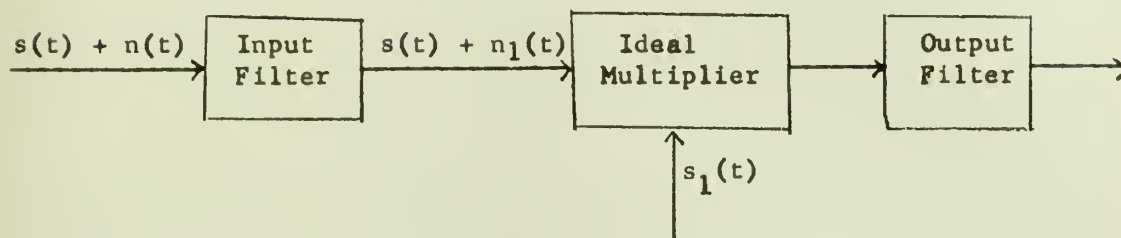
$$S = \frac{b}{\cos \theta} = \frac{E}{\sqrt{2} \sigma_f} = \frac{E}{\sqrt{K}}$$

$$\omega_1(\theta) = \frac{1}{\pi} e^{-S^2} \frac{1}{S} \frac{F}{\cos \theta}$$



APPENDIX C

Analysis of an FM wave through the synchronous detector



The input filter has the transfer function

$$\begin{aligned}
 H_1(f) &= \frac{1}{\sqrt{1 + \left(\frac{f - f_c}{\alpha}\right)^2}} && \text{for } f \geq 0 \\
 &= \frac{1}{\sqrt{1 + \left(\frac{f + f_c}{\alpha}\right)^2}} && \text{for } f \leq 0
 \end{aligned}$$

where f_c = input filter center frequency

$\alpha = \frac{1}{2}$ power bandwidth

$\alpha \ll f_c$

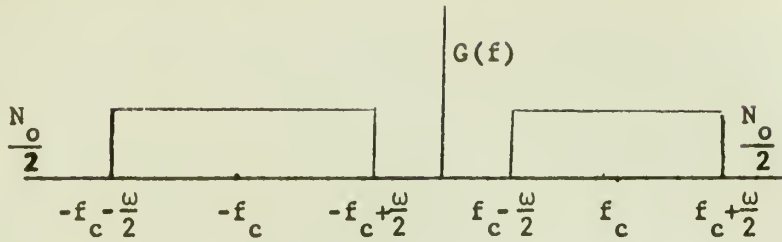
The output bandpass filter has the transfer function

$$\begin{aligned}
 H_o(f) &= \frac{1}{\sqrt{1 + \left(\frac{f - f_m}{\beta}\right)^2}} && \text{for } f \geq 0 \\
 &= \frac{1}{\sqrt{1 + \left(\frac{f + f_m}{\beta}\right)^2}} && \text{for } f \leq 0
 \end{aligned}$$

The input noise $n(t)$ is assumed to be additive white gaussian noise centered about f_c and band limited to W cycles with power spectral density of N_0 watts/cycle.



The spectrum of the noise is thus:



It is assumed that the intelligence part of the received signal $s(t)$ is passed completely by the input filter unattenuated. Since $W \gg \alpha$ then the noise will be perturbed in some manner and will be expressed as $n_1(t)$. The input to the multiplier is thus: $s(t) + n_1(t)$.

$s(t) = \sqrt{2} A [\cos \omega_c t + mf \cos \omega_m t \sin \omega_c t]$ since the FM wave has a small modulation index and

$$s_1(t) = \sqrt{2} C \sin (\omega_c t + \theta)$$

The multiplier output is then

$$\begin{aligned} e_{om} &= \left\{ \sqrt{2} A [\cos \omega_c t + mf \cos \omega_m t \sin \omega_c t] + n_1(t) \right\} \left\{ \sqrt{2} C \sin (\omega_c t + \theta) \right\} \\ &= 2AC \cos \omega_c t \sin(\omega_c t + \theta) + 2AC mf \cos \omega_m t \sin \omega_c t \sin(\omega_c t + \theta) \\ &\quad + \sqrt{2} C \sin(\omega_c t + \theta) n_1(t) \\ &= 2AC \cos \omega_c t [\sin \omega_c t \cos \theta + \cos \omega_c t \sin \theta] \\ &\quad + 2 AC mf \cos \omega_m t \sin \omega_c t [\sin \omega_c t \cos \theta + \cos \omega_c t \sin \theta] \\ &\quad + \sqrt{2} C \sin(\omega_c t + \theta) n_1(t) \\ &= 2AC \left[\frac{1}{2} \sin 2\omega_c t \cos \theta + \frac{\sin \theta}{2} + \frac{\sin \theta}{2} \cos 2\omega_c t + \frac{mf}{2} \cos \omega_m t \cos \theta \right. \\ &\quad \left. + \frac{mf}{2} \cos \omega_m t \cos \theta \sin 2\omega_c t + \frac{mf}{2} \cos \omega_m t \sin \theta \sin 2\omega_c t \right] \\ &\quad + \sqrt{2} C \sin(\omega_c t + \theta) n_1(t) \end{aligned}$$



Neglecting the $2\omega_c t$ terms since they are filtered out

$$e_{om} = AC [\sin \theta + mf \cos \omega_m t \cos \theta] + \sqrt{2} C \sin(\omega_c t + \theta) n_1(t)$$

$$e_{om} = AC \sin \theta + (mf AC \cos \theta) \cos \omega_m t + \sqrt{2} C \sin(\omega_c t + \theta) n_1(t)$$

The signal out of the bandpass filter will be $(mf AC \cos \theta) \cos \omega_m t$ with signal power $\frac{1}{2} (mf AC \cos \theta)^2$.

It is desired to find the noise power at output of the bandpass filter. This can most easily be found from the correlation function.

$$\text{Let } e(t) = n_1(t) \sqrt{2} C \cos(\omega_c t + \theta)$$

$$\begin{aligned} \phi_{om}(\tau) &= \overline{e(t) e(t + \tau)} \\ &= 2C^2 \overline{n_1(t) n_1(t + \tau) \cos(\omega_c t + \theta) \cos \omega_c[(t + \tau) + \theta]} \end{aligned}$$

Since $n_1(t)$ and $\cos \omega_c t$ are independent, this may be written

$$\phi_{om}(\tau) = 2C^2 \overline{n_1(t) n_1(t + \tau)} \times \overline{\cos(\omega_c t + \theta) \cos[\omega_c(t + \tau) + \theta]}$$

$$\phi_{om}(\tau) = [\phi_{nn}(\tau) \phi_{cc}(\tau)]$$

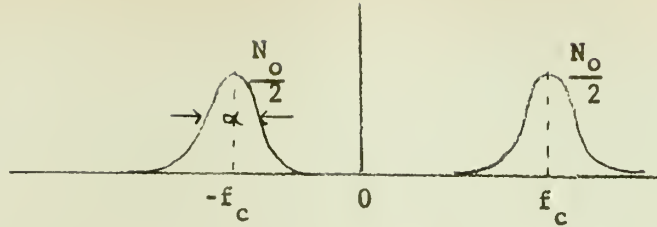
By the convolution theorem

$$G_{om} = \mathcal{F}\{\phi_{om}(\tau)\} = \int_{-\infty}^{\infty} G_{n_1}(\sigma) G_c(f - \sigma) d\sigma$$

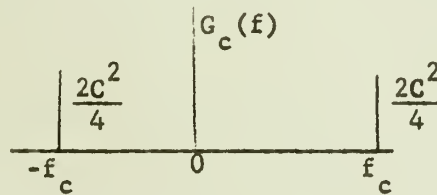
$$\begin{aligned} \text{Now } G_{n_1}(f) &= \frac{N_o}{2} |H_1(f)|^2 = \frac{N_o}{2} \times \frac{1}{1 + \left(\frac{f - f_c}{\alpha}\right)^2} \quad f \geq 0 \\ &= \frac{N_o}{2} \times \frac{1}{1 + \left(\frac{f + f_c}{\alpha}\right)^2} \quad f \leq 0 \end{aligned}$$

A plot of $G_{n_1}(f)$ is as follows:

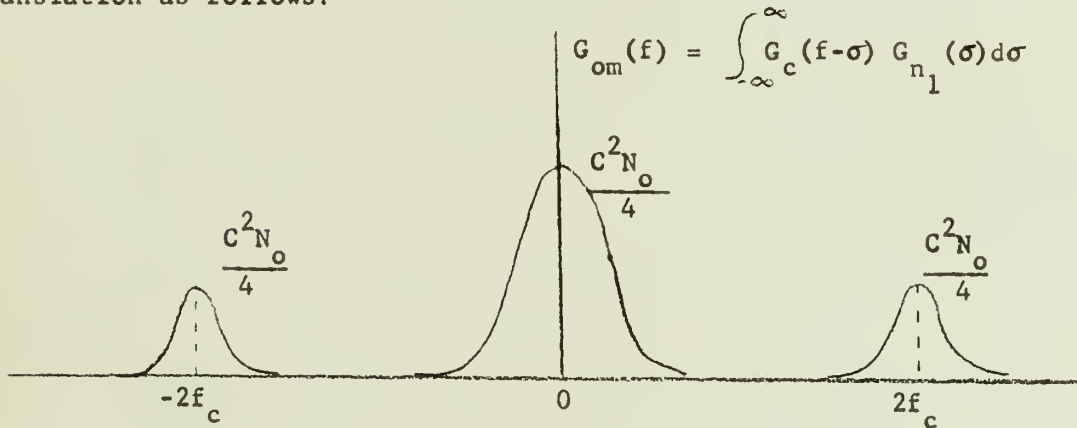




$$G_c(f) = \frac{2C^2}{4} \left[\sigma(f - f_c) + \sigma(f + f_c) \right] \text{ and is plotted below}$$



The graphical integration of the Fourier transform of ϕ_{om} or the convolution product of $\phi_{nn}(f) \phi_{cc}(f)$ essentially amounts to a frequency translation as follows:



The low frequency portion of this spectrum may be written

$$G_{om}(f) = \frac{C^2 N_o}{2} \times \frac{1}{1 + \left(\frac{f}{\alpha}\right)^2}$$

The spectrum at the output of the bandpass filter is

$$G_o(f) = G_{om}(f) \left| H_o(f) \right|^2$$



$$\begin{aligned}
G_o(f) &= G_{om}(f) |H_o(f)|^2 \\
&= \frac{C_{N_o}^2}{2} \times \frac{1}{1 + \left(\frac{f}{\alpha}\right)^2} \times \frac{1}{1 + \left(\frac{f - f_m}{\beta}\right)^2} \quad \text{for } f \geq 0 \\
&= \frac{C_{N_o}^2}{2} \times \frac{1}{1 + \left(\frac{f}{\alpha}\right)^2} \times \frac{1}{1 + \left(\frac{f + f_m}{\beta}\right)^2} \quad \text{for } f \leq 0
\end{aligned}$$

Since $\beta \ll \alpha$ then $G_{om}(f)$ may be considered as a constant with a value determined by f_m and since $f_m \ll \alpha$ then it is essentially $\frac{C_{N_o}^2}{2}$. The output noise is then

$$G_o(f) = \frac{C_{N_o}^2}{2} \left[\int_{-\infty}^0 \frac{df}{1 + \left(\frac{f + f_m}{\beta}\right)^2} + \int_0^{\infty} \frac{df}{1 + \left(\frac{f - f_m}{\beta}\right)^2} \right]$$

$$\begin{aligned}
\text{Let } u &= \frac{f + f_m}{\beta} & \text{when } f = 0 \quad u &= \frac{f_m}{\beta} \\
du &= \frac{df}{\beta} & f = -\infty \quad u &= -\infty
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\frac{f_m}{\beta}} \frac{\beta du}{1 + u^2} &= \beta \tan^{-1} u \Big|_{-\infty}^{\frac{f_m}{\beta}} = \beta \left[\tan^{-1} \frac{f_m}{\beta} - \beta \tan^{-1} \infty \right] \\
&\cong \beta \pi \quad \text{for } \beta \ll f_m
\end{aligned}$$

The second integral can be evaluated in a similar manner. The output noise power is then

$$G_o(f) = \frac{C_{N_o}^2}{2} [2\pi\beta] = \beta \pi N_o C^2$$

The ratio of the output signal power to the output noise power is

$$\left(\frac{S}{N}\right)_{\text{out FM}} = \frac{\frac{1}{2} (m_f AC \cos \theta)^2}{\pi \beta N_o C^2} = \left[\frac{m_f^2 \cos^2 \theta}{2\pi} \right] \left[\frac{A^2}{\beta N_o} \right]$$



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A comparison between frequency modulatio



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